Maximum-Norm Resolvent Estimates and Stability for Parabolic Finite Element Problems

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In this lecture we give a survey of work over the last decades on stability and smoothing estimates in maximum-norm of spatially semidiscrete finite element approximations of a model parabolic equation, and related such estimates for the resolvent of the corresponding discrete elliptic operator. We end with a short discussion of stability of fully discrete time stepping methods. For simplicity of presentation we restrict ourselves here to piecewise linear finite elements in two space dimensions, even though several of the results described are valid in greater generality. Several of the results are obtained in collaboration with Michel Crouzeix, to whom this lecture is dedicated.

Consider the initial-value problem

$$u_t - \Delta u = 0 \quad \text{in } \Omega \quad \text{and } u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{in } \Omega. \tag{1}$$

where Ω is a domain in $\mathbb{R}^2$, and denote by $E(t)$ the solution operator for this problem defined by $u(t) = E(t)v$. It is a special case of a result of Stewart [9] that if $\partial\Omega$ is smooth, then $E(t)$ is an analytic semigroup on $\mathcal{C}_0(\bar\Omega) = \{v \in \mathcal{C}(\bar\Omega) : \ v = 0 \text{ on } \partial\Omega\}$, generated by Δ. This follows from the resolvent estimate

$$\|(\lambda I + \Delta)^{-1} v\|_{\mathcal{C}} \le \frac{C}{1+|\lambda|} \|v\|_{\mathcal{C}}, \quad \text{for } \lambda \notin \Sigma_\delta = \{\lambda : \ |\arg \lambda| \le \delta\},$$

where $\|v\|_{\mathcal{C}} = \sup_{x\in\Omega} |v(x)|$ and where $\delta \in (0, \frac12\pi)$ is arbitrary. In addition to the stability estimate $\|E(t)v\|_{\mathcal{C}} \le \|v\|_{\mathcal{C}}$, which follows by the maximum-principle, this entails the smoothing estimate

$$\|E'(t)v\|_{\mathcal{C}} \le \frac{C}{t} \|v\|_{\mathcal{C}}, \quad \text{for } v \in \mathcal{C}_0(\bar\Omega).$$

Such a result is valid also under lesser regularity requirements on $\partial\Omega$.

We shall discuss spatially semidiscrete and fully discrete approximation of (1) based on continuous, piecewise linear finite elements, defined on a family of regular triangulations $\mathcal{T}_h = \{\tau\}$ of $\bar\Omega$ into closed triangles τ. We set $h = \max_{\tau\in\mathcal{T}_h} \operatorname{diam}(\tau)$, and assume that the interior Ω_h of $\cup\{\tau : \tau \in \mathcal{T}_h\} \subseteq \Omega$ is a subset of Ω. We associate with $\mathcal{T}_h$ the finite dimensional space

$$S_h = \{\chi \in \mathcal{C}(\bar\Omega) : \ \chi|_\tau \text{ linear for } \tau \in \mathcal{T}_h, \ \chi = 0 \text{ on } \partial\Omega \cup (\Omega \setminus \Omega_h)\}.$$

The semidiscrete finite element problem associated with (1) is then to find $u_h(t) \in S_h$ for $t > 0$ such that

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) = 0 \quad \text{for } \chi \in S_h, \ t > 0, \quad \text{with } u_h(\cdot, 0) = v_h, \quad \text{where } (v, w) = \int_\Omega v\, \overline{w}\, dx. \tag{2}$$

With $\Delta_h : S_h \to S_h$ defined by $-(\Delta_h\psi, \chi) = (\nabla\psi, \nabla\chi), \ \forall\, \psi, \chi \in S_h$, this may also be written as

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t > 0, \quad \text{with } u_h(0) = v_h.$$

The solution operator of this problem, defined by $u_h(t) = E_h(t)v_h$, is the semigroup $E_h(t) = e^{\Delta_h t}$ in S_h generated by Δ_h. The issue is then to show that this semigroup is analytic in S_h, equipped with the maximum-norm, and this may be expressed either as a resolvent estimate for $-\Delta_h$ or as the stability and a smoothing property of $E_h(t)$. We remark that a maximum-principle does not hold for (2) and that $E_h(t)$ is not a contraction, see [10].

In Schatz, Thomée and Wahlbin [7] it was thus shown, by weighted norm energy arguments, that in the case of a convex domain Ω with smooth boundary, and for quasiuniform triangulations,

$$\|E_h(t)v_h\|_{\mathcal{C}} + t\|E_h'(t)v_h\|_{\mathcal{C}} \le C\ell_h \|v_h\|_{\mathcal{C}}, \quad \text{for } v_h \in S_h, \quad \text{where } \ell_h = \max(1, \log(1/h)). \tag{3}$$

By semigroup theory this shows the resolvent estimate

$$\|(\lambda I + \Delta_h)^{-1} v_h\|_{\mathcal{C}} \le \frac{C\ell_h^2}{1+|\lambda|} \|v_h\|_{\mathcal{C}}, \quad \text{for } \lambda \notin \Sigma_{\delta_h}, \quad \text{where } \delta_h = \tfrac12\pi - c\ell_h^{-2}. \tag{4}$$

In Schatz, Thomée, and Wahlbin [8] the logarithmic factor in (3) was removed, which implies that the resolvent estimate (4) holds without a logarithmic factor, and for $\lambda \notin \Sigma_\delta$, for some $\delta \in (0, \frac{1}{2}\pi)$, independent of h. In Bakaev, Thomée and Wahlbin [1] a direct proof was given that, for $\partial\Omega$ smooth, this resolvent estimate holds for any angle $\delta \in (0, \frac{1}{2}\pi)$. In Chatzipantelidis, Lazarov, Thomée, and Wahlbin [6] such a resolvent estimate, with a logarithmic factor, was shown when Ω is a plane polygonal domain, which may be nonconvex.

In all these results quoted the family of triangulations was required to be quasiuniform, which is a somewhat undesirable restriction. A first attempt to weaken this requirement was made in Crouzeix and Thomée [5] where a resolvent estimate of the desired type, with a logarithmic factor, was shown for a modified discrete Laplacian, defined by

$$-(\Delta_h \psi, \chi)_h = (\nabla\psi, \nabla\chi), \quad \forall \psi, \chi \in S_h,$$

where $(\cdot,\cdot)_h$ denotes a simple quadrature approximation of the L_2-inner product, and for triangulations of Delaunay type, not required to be quasiuniform. This choice corresponds to the so-called lumped mass modification of the semidiscrete problem (2) defined by

$$(u_{h,t}, \chi)_h + (\nabla u_h, \nabla\chi) = 0 \quad \text{for } \chi \in S_h,\ t > 0, \quad \text{with } u_h(\cdot, 0) = v_h \quad \text{in } \Omega.$$

For this problem a maximum principle holds, and the solution operator $\bar{E}_h(t) = e^{\bar{\Delta}_h h}$ is a contraction with respect to the maximum-norm.

We return to the standard semidiscrete problem (2). Given $\tau_0 \in \mathcal{T}_h$, we let $Q_j(\tau_0)$ denote the set of triangles which are "j triangles away from τ_0", and by $n_j(\tau_0)$ the number of triangles in $Q_j(\tau_0)$. We now make the assumption that the family $\{\mathcal{T}_h\}$ of triangulations satisfies, with some $\alpha \geq 1$ and $\beta \geq 1$,

$$h_\tau / h_{\tau_0} \leq C\alpha^j \quad \text{and } n_j(\tau) \leq C\beta^j, \quad \text{for } \tau \in Q_j(\tau_0), \quad j \geq 1, \quad \text{for all } \tau_0 \in \mathcal{T}_h. \tag{5}$$

For quasiuniform triangulations this holds with $\alpha = 1$ and β any number > 1, and if $\alpha > 1$ we may choose $\beta = \alpha^4$. Under these assumptions it was shown in Bakaev, Crouzeix and Thomée [2] that if (5) holds and if $\alpha^2\beta\gamma < 1$, with $\gamma = .318$, then, for any fixed $\delta \in (0, \frac{1}{2}\pi)$, we have, with $h_{min} = \min_{\tau\in\mathcal{T}_h} h_\tau$,

$$\|(\lambda I + \Delta_h)^{-1}\chi\|_{\mathcal{C}} \leq \frac{C\ell_h^{1/2}}{1 + |\lambda|}, \quad \forall \chi \in S_h, \quad \lambda \notin \Sigma_\delta. \quad \text{where } \ell_h = \max(1, \log(1/h_{min})), \tag{6}$$

With $\beta = \alpha^4$, the condition requires $\alpha < \gamma^{-1/6} = (.318)^{-1/6} \approx 1.21$, which permits a substantial degree of nonquasiuniformity. The proof of this result depends on an exponential decay property of the L_2-projection P_h, which states that if $supp\, v \subset \tau_0$, then $\|P_h v\|_{L_2(\tau)} \leq C\gamma^j \|v\|_{L_2}$ for $\tau \in Q_j(\tau_0)$, see Crouzeix and Thomée [4]. It follows from (6) that, under our present assumptions on $\mathcal{T}_h$, the solution operator $E_h(t)$ of (2) satisfies the stability and smoothing estimate (3), with ℓ_h replaced by $\ell_h^{1/2}$.

We now turn to fully discrete schemes for (1) which we obtain by time stepping in the spatially semidiscrete equation (2). It is convenient to treat the time stepping in a Banach space framework, and we consider thus an initial value problem of the form

$$u' + Au = 0, \quad \text{for } t > 0, \quad \text{with } u(0) = v, \tag{7}$$

in a complex Banach space $\mathcal{B}$ with norm $\|\cdot\|$. We assume that A is a closed, densely defined linear operator, such that its resolvent satisfies, for some $\delta \in (0, \frac{1}{2}\pi)$,

$$\|(\lambda I - A)^{-1}\| \leq M|\lambda|^{-1}, \quad \text{for } \lambda \notin \Sigma_\delta, \quad \text{with } M \geq 1.$$

We recall that $-A$ is then the infinitesimal generator of an analytic semigroup

$$E(t) = e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(\lambda I - A)^{-1}\, d\lambda, \quad \text{for } t \geq 0,$$

which is the solution operator of (7), and where, e.g., $\Gamma = \{\lambda;\ |\arg\lambda| = \psi \in (\delta, \frac{1}{2}\pi)\}$.

We shall now discuss discretization in time of (7). Letting k denote the time step and $t_n = nk$, and letting $r(\lambda)$ be a rational function which is bounded on Σ_ψ, we define the approximation U_k^n of $u(t_n) = E(t_n)v$ by the recursion formula

$$U_k^{n+1} = E_k U_k^n, \quad \text{for } n \geq 0, \quad \text{where } E_k = r(kA), \quad \text{with } U_k^0 = v.$$

We may thus write $U_k^n = E_k^n v$. It was shown in Crouzeix, Larsson, Piskarev and Thomée [3] that if $r(\lambda)$ is $A(\theta)-$stable, with $\theta \in (\delta, \frac{1}{2}\pi]$, then

$$\|U_k^n\| = \|E_k^n v\| \le CM\|v\|.$$

The proof uses that, for any rational function $r(\lambda)$, bounded in Σ_ψ, we have, with suitable Γ,

$$r(A) = r(\infty)I + \frac{1}{2\pi i}\int_\Gamma r(\lambda)(\lambda I - A)^{-1}\, d\lambda.$$

As an example, for the Crank-Nicolson method, corresponding to the A-stable rational function $r(\lambda) = (1+\frac{1}{2}\lambda)^{-1}(1-\frac{1}{2}\lambda)$, we obtain, under the assumption that (6) holds, that for the fully discrete solution $U_{kh}^n = E_{kh}^n v_h$, with $E_{kh} = r(-k\Delta_h)$, we have $\|U_{kh}^n\|_{\mathcal{C}} \le C\ell_h^{1/2}\|v_h\|_{\mathcal{C}}$.

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