Electromagnetic imaging: Representation formulas for the field perturbations caused by low volume fraction inhomogeneities

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\[
\begin{cases}
\text{div} \,(A(x) \, \text{grad} u) = \nabla \cdot (A(x) \nabla u) = 0 \quad \text{in } \Omega \subset IR^n \\
(A \, \text{grad} u) \cdot \nu = (A \nabla u) \cdot \nu = \phi \quad \text{on } \partial \Omega.
\end{cases}
\]

\begin{align*}
(A(x)\xi) \cdot \xi \geq c_0|\xi|^2 , \quad & \int_{\partial \Omega} \phi = 0 , \quad \nu \text{ is outward normal}.
\end{align*}
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\end{align*}
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\[(A(x)\xi) \cdot \xi \geq c_0|\xi|^2, \quad \int_{\partial \Omega} \phi = 0, \quad \nu \text{ is outward normal}.
\]

**Define :** \( \Lambda_A[\phi] = u|_{\partial \Omega} \) as a bounded linear operator from \( H^{-1/2}(\partial \Omega) \) into \( H^{1/2}(\partial \Omega) \).
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\((A(x)\xi) \cdot \xi \geq c_0 |\xi|^2, \quad \int_{\partial\Omega} \phi = 0, \quad \nu \text{ is outward normal}.\)

**Define**: \(\Lambda_A[\phi] = u|_{\partial\Omega}\) as a bounded linear operator from \(H^{-1/2}(\partial\Omega)\) into \(H^{1/2}(\partial\Omega)\).

**Question**: (Calderon, 1980) does knowledge of \(\Lambda_A\) determine \(A(\cdot)\) ?
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**Define** : \( \Lambda_A[\phi] = u|_{\partial \Omega} \) as a bounded linear operator from \( H^{-1/2}(\partial \Omega) \) into \( H^{1/2}(\partial \Omega) \).

**Question** : (Calderon, 1980) does knowledge of \( \Lambda_A \) determine \( A(\cdot) \) ?

**Answer 1**: – Yes, if \( A \) is isotropic, i.e., if \( A(x) = \gamma(x)I \).

Astala-Paivarinta (2003) \([L^\infty , \ n = 2]\), Nachman (1996)\([C^\infty , \ n = 2]\),
Sylvester-Uhlmann (1987) \([C^\infty , \ n \geq 3]\), Kohn-Vogelius (1985)

[piecewise analytic, \( n \geq 2 \)].
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**Question:** (Calderon, 1980) does knowledge of \(\Lambda_A\) determine \(A(\cdot)\)?

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**Answer 2:** – Only very partially (up to a “pullback” by a diffeomorphism) if \(A\) is anisotropic. Sylvester (1990), Lee-Uhlmann (1987)[“positive”], Kohn-Vogelius, Tartar (1987)[“negative”].
RELATED WORK BY: Alessandrini, Bryan, Cheney, Cherkaev, Colton, Dobson, Druskin, Friedman, Isaacson, Isakov, Kang, Kress, Moskow, Santosa, Seo ...
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Area of significant practical interest:

Internal conductivity inhomogeneities of low volume fraction
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The inhomogeneities could represent: tumors (in medical applications), defects (in materials), or anti-personnel mines .... just to mention a few applications.
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\[ \gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & \text{in } \Omega \setminus \omega_\epsilon \\ \gamma_1(x) & \text{in } \omega_\epsilon \end{cases} \]

\(\omega_\epsilon \subset K_0 \subset \Omega\)

\(0 < \gamma_0(x), \gamma_1(x) < \infty\)

\[ \begin{cases} \nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \phi & \text{on } \partial \Omega. \end{cases} \]
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\gamma_\epsilon(x) = \begin{cases} 
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\nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\
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\end{cases}
\]

**Goal:** Find an asymptotic expression for \((u_\epsilon - u_0)|_{\partial \Omega}\) that can be used to determine \(\omega_\epsilon\) (for \(|\omega_\epsilon|\) small).
General Representation Formula

After the extraction of a subsequence:

\[ \forall y \in \partial \Omega, \ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega + o(|\omega_\epsilon|) \]
General Representation Formula

After the extraction of a subsequence:

\[ \forall y \in \partial \Omega, \; u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega \nabla \cdot (\gamma_0 \nabla N(x, y)) + o(|\omega_\epsilon|) \]

\[ \frac{\partial N}{\partial x_i}(x, y) \]

\( N(x, y) \) is the Neumann function for \( \nabla \cdot (\gamma_0 \nabla ) \):

\[ \nabla \cdot (\gamma_0 \nabla N(x, y)) = \delta_y \text{ in } \Omega \]

\[ \gamma_0(x) \frac{\partial N}{\partial \nu_x} = \frac{1}{|\partial \Omega|} \text{ on } \partial \Omega. \]
General Representation Formula

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\[ \forall y \in \partial \Omega, \ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|) \]

\( N(x, y) \) is the Neumann function for \( \nabla \cdot (\gamma_0 \nabla \cdot \cdot \cdot ) \):

\[ \nabla_x \cdot (\gamma_0 \nabla_x N(x, y)) = \delta_y \text{ in } \Omega \]

\[ \gamma_0(x) \frac{\partial N}{\partial \nu_x} = \frac{1}{|\partial \Omega|} \text{ on } \partial \Omega. \]

The probability measure \( \mu = \lim |\omega_\epsilon| \rightarrow 0 \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \) converges weak* in the dual of \( C^0(\overline{\Omega}) \).
General Representation Formula

After the extraction of a subsequence:

$$\forall y \in \partial \Omega, \ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)$$

\(N(x, y)\) is the Neumann function for \(\nabla \cdot (\gamma_0 \nabla )\):

$$\nabla_x \cdot (\gamma_0 \nabla_x N(x, y)) = \delta_y \text{ in } \Omega$$

$$\gamma_0(x) \frac{\partial N}{\partial \nu_x} = \frac{1}{|\partial \Omega|} \text{ on } \partial \Omega.$$  

The probability measure \(\mu = \lim_{|\omega_\epsilon| \to 0} \frac{1}{|\omega_\epsilon|} \omega_\epsilon\) converges weak* in the dual of \(C^0(\overline{\Omega})\).

\(M\) is a matrix valued function in \(L^2(\Omega, d\mu)\). The values of \(M\) are symmetric, positive definite matrices.
Definition of $M$

$$
\int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla v dx =
$$
Definition of $M$

$$
\int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla v d x = \int_{\Omega} (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v
$$
Definition of $M$

$$\int_{\Omega} \gamma_0 \nabla (u_\varepsilon - u_0) \nabla v dx = \int_{\Omega} (\gamma_0 - \gamma_\varepsilon) \nabla u_\varepsilon \nabla v$$

$$= \int_{\omega_\varepsilon} (\gamma_0 - \gamma_1) \nabla u_\varepsilon \nabla v$$
Definition of $M$

$$\int_\Omega \gamma_0 \nabla (u_\epsilon - u_0) \nabla v dx = \int_\Omega (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v = \int_{\omega_\epsilon} (\gamma_0 - \gamma_1) \nabla u_\epsilon \nabla v$$

Permit $v(\cdot) = N(\cdot, y)$ – by approximation; for $y \in \Omega \setminus K_0$, $(u_\epsilon - u_0)$ is smooth.
Definition of $M$

\[
\int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla v \, dx = \int_{\Omega} (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v
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Permit $v(\cdot) = N(\cdot, y)$ – by approximation; for $y \in \Omega \setminus K_0$, $(u_\epsilon - u_0)$ is smooth.

\[-w(y) + \oint_{\partial \Omega} w(x) \, d\sigma_x = \int_{\Omega} \gamma_0 \nabla_x w(x) \nabla_x N(x, y) \, dx \text{ for all } w\]
Definition of $M$

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\int_{\Omega} \gamma_0 \nabla (u_{\epsilon} - u_0) \nabla v \, dx \quad = \quad \int_{\Omega} (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v \\
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\leadsto \quad (u_\epsilon - u_0)(y) \quad = \quad - \int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla N(x, y) \, dx
\]
Definition of $M$

$$\int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla v \, dx = \int_{\Omega} (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v$$

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Permit $v(\cdot) = N(\cdot, y)$ - by approximation; for $y \in \Omega \setminus K_0$, $(u_\epsilon - u_0)$ is smooth.

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$$\Rightarrow (u_\epsilon - u_0)(y) = - \int_{\Omega} \gamma_0 \nabla (u_\epsilon - u_0) \nabla N(x, y) \, dx$$

$$= \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla u_\epsilon \nabla N(x, y) \, dx$$
Definition of $M$

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\int_\Omega \gamma_0 \nabla (u_\epsilon - u_0) \nabla v \, dx = \int_\Omega (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v \, dx = \int_{\omega_\epsilon} (\gamma_0 - \gamma_1) \nabla u_\epsilon \nabla v \, dx
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\[\leadsto (u_\epsilon - u_0)(y) = -\int_\Omega \gamma_0 \nabla (u_\epsilon - u_0) \nabla N(x,y) \, dx = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \nabla u_\epsilon \nabla N(x,y) \, dx\]
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\int_{\Omega} \gamma_0 \nabla (u_\varepsilon - u_0) \nabla v dx = \int_{\Omega} (\gamma_0 - \gamma_\varepsilon) \nabla u_\varepsilon \nabla v
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$$

for all $w$

$$
\sim \quad (u_\varepsilon - u_0)(y) = -\int_{\Omega} \gamma_0 \nabla (u_\varepsilon - u_0) \nabla N(x, y) dx
$$

$$
= |\omega_\varepsilon| \int_{\Omega} (\gamma_1 - \gamma_0) \frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon} \nabla u_\varepsilon \nabla N(x, y) dx
$$

$$
\frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon} \nabla u_\varepsilon dx \rightarrow M \nabla u_0 d\mu
$$
Definition of $M$

Next we prove

$$\left\| \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \nabla u_\epsilon \right\|_{L^1(\Omega)} \leq C$$
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This permits to prove

$$\frac{1}{|\omega_{\epsilon}|} 1_{\omega_{\epsilon}} \frac{\partial u_{\epsilon}}{\partial x_i} \rightharpoonup M_{ij} \frac{\partial u_0}{\partial x_j} d\mu \text{ (up to a subseq.)}$$
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\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_i} \rightharpoonup M_{ij} \frac{\partial u_0}{\partial x_j} d\mu \text{ (up to a subseq.)}
$$

$L^2$ estimate:

$$
\int_\Omega \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v dx = \int_\Omega (\gamma_0 - \gamma_1) 1_{\omega_\epsilon} \nabla u_0 \nabla v dx.
$$
Definition of $M$

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Thus

$$\| \nabla (u_\epsilon - u_0) \|^2_{L^2(\Omega)} \leq C \left( \int_\Omega 1_{\omega_\epsilon} |\nabla u_0|^2 \right)^{1/2} \| \nabla (u_\epsilon - u_0) \|_{L^2(\Omega)},$$
Definition of $M$

Next we prove

$$\left\| \frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon} \nabla u_\varepsilon \right\|_{L^1(\Omega)} \leq C$$

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Thus

$$\| \nabla (u_\varepsilon - u_0) \|^2_{L^2(\Omega)} \leq C \left( \int_{\Omega} 1_{\omega_\varepsilon} |\nabla u_0|^2 \right)^{\frac{1}{2}} \| \nabla (u_\varepsilon - u_0) \|_{L^2(\Omega)},$$

and

$$\| \nabla (u_\varepsilon - u_0) \|_{L^2(\Omega)} \leq C |\omega_\varepsilon|^{\frac{1}{2}} \quad (u_0 \text{ is smooth}).$$
Definition of $M$

$L^1$ estimate:

$$\int_{\omega_\varepsilon} \frac{1}{|\omega_\varepsilon|} |\nabla u_\varepsilon| \, dx \leq$$
Definition of $M$

$L^1$ estimate:

$$\int_{\omega_{\epsilon}} \frac{1}{|\omega_{\epsilon}|} |\nabla u_{\epsilon}| \, dx \leq \frac{1}{|\omega_{\epsilon}|} \int_{\omega_{\epsilon}} |\nabla u_{\epsilon} - \nabla u_0| \, dx + \frac{1}{|\omega_{\epsilon}|} \int_{\omega_{\epsilon}} |\nabla u_0| \, dx$$
Definition of $M$

$L^1$ estimate:

$$\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} |\nabla u_\epsilon| \, dx \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_\epsilon - \nabla u_0| \, dx + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_0| \, dx$$

$$\leq \frac{1}{|\omega_\epsilon|} \left( \int_{\omega_\epsilon} 1 \right)^{\frac{1}{2}} \|\nabla u_\epsilon - \nabla u_0\|_{L^2(\Omega)} + C$$
Definition of $M$

$L^1$ estimate:

\[
\int_{\omega_\varepsilon} \frac{1}{|\omega_\varepsilon|} \left| \nabla u_\varepsilon \right| \, dx \leq \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \left| \nabla u_\varepsilon - \nabla u_0 \right| \, dx + \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \left| \nabla u_0 \right| \, dx
\]

\[
\leq \frac{1}{|\omega_\varepsilon|} \left( \int_{\omega_\varepsilon} 1 \right)^{\frac{1}{2}} \left\| \nabla u_\varepsilon - \nabla u_0 \right\|_{L^2(\Omega)} + C
\]

\[
\leq C.
\]
Definition of $M$

$L^1$ estimate:

$$\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} |\nabla u_\epsilon| \, dx \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_\epsilon - \nabla u_0| \, dx + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_0| \, dx$$

$$\leq \frac{1}{|\omega_\epsilon|} \left( \int_{\omega_\epsilon} 1 \right)^{\frac{1}{2}} \|\nabla u_\epsilon - \nabla u_0\|_{L^2(\Omega)} + C$$

$$\leq C.$$ 

Bounded, so $\frac{1}{|\omega_\epsilon|} \omega_\epsilon \frac{\partial u_\epsilon}{\partial x_i} \rightarrow$ something
Definition of $M$

$L^1$ estimate:

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\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} |\nabla u_\epsilon| \, dx \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_\epsilon - \nabla u_0| \, dx + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_0| \, dx
$$

$$
\leq \frac{1}{|\omega_\epsilon|} \left( \int_{\omega_\epsilon} 1 \right)^{\frac{1}{2}} \|\nabla u_\epsilon - \nabla u_0\|_{L^2(\Omega)} + C
$$

$$
\leq C.
$$

Bounded, so

$$
\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_i} \to M_{ij} \frac{\partial u_0}{\partial x_j} \, d\mu.
$$
Definition of $M$

$L^1$ estimate:
\[
\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} |\nabla u_\epsilon| \, dx \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_\epsilon - \nabla u_0| \, dx + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_0| \, dx
\]
\[
\leq \frac{1}{|\omega_\epsilon|} \left( \int_{\omega_\epsilon} 1 \right)^{\frac{1}{2}} \| \nabla u_\epsilon - \nabla u_0 \|_{L^2(\Omega)} + C
\]
\[
\leq C.
\]

Bounded, so \( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \partial u_\epsilon \partial x_i \rightarrow M_{ij} \frac{\partial u_0}{\partial x_j} \, d\mu \).

Formally, $M_{ij}$ is obtained by taking $\frac{\partial u_0}{\partial x_k} = \delta_{jk}$, i.e., $u_0 = x_j + cst.$
Definition of $M$

$L^1$ estimate:

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\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} |\nabla u_\epsilon| \, dx \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_\epsilon - \nabla u_0| \, dx + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} |\nabla u_0| \, dx
$$

$$
\leq \frac{1}{|\omega_\epsilon|} \left( \int_{\omega_\epsilon} 1 \right)^{\frac{1}{2}} \|\nabla u_\epsilon - \nabla u_0\|_{L^2(\Omega)} + C
$$

$$
\leq C.
$$

Bounded, so $\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_i} \rightharpoonup M_{ij} \frac{\partial u_0}{\partial x_j} d\mu$.

Formally, $M_{ij}$ is obtained by taking $\frac{\partial u_0}{\partial x_k} = \delta_{jk}$, i.e., $u_0 = x_j + \text{cst}$.

But, notice that $v_0^j = x_j + \text{cst}$ is not a solution to

$$
\begin{cases}
\nabla \cdot (\gamma_0 \nabla v_0) = 0 & \text{in } \Omega \\
\gamma_0 \frac{\partial v_0}{\partial \nu} = \psi & \text{on } \partial \Omega.
\end{cases}
$$

unless $\gamma_0$ is a constant!
Definition of $M$

Instead, construct

\[
\begin{cases}
\nabla \cdot \left( \gamma_\varepsilon \nabla v^j_\varepsilon \right) = \nabla \cdot \left( \gamma_0 \nabla v^j_0 \right) & \text{in } \Omega \\
\gamma_\varepsilon \frac{\partial v^j_\varepsilon}{\partial \nu} = \gamma_0 \frac{\partial v^j_0}{\partial \nu} & \text{on } \partial \Omega, \\
v^j_0 = x_j + \text{cst},
\end{cases}
\]
Definition of $M$

Instead, construct

$$\begin{cases} 
\nabla \cdot (\gamma_\varepsilon \nabla v_\varepsilon^j) = \nabla \cdot (\gamma_0 \nabla v_0^j) & \text{in } \Omega \\
\gamma_\varepsilon \frac{\partial v_\varepsilon^j}{\partial \nu} = \gamma_0 \frac{\partial v_0^j}{\partial \nu} & \text{on } \partial \Omega, \\
v_0^j = x_j + \text{cst},
\end{cases}$$

and define

$$\frac{1}{|\omega_\varepsilon|} \omega_\varepsilon \frac{\partial v_\varepsilon^j}{\partial x_k} \overset{\text{def}}{=} M_{jk} d\mu.$$
To prove \[ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \, dx \to M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \]

\[ \frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j \, dx \]

\[ \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j \, dx \]
To prove \( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \partial u_\epsilon \partial x_j \, dx \rightarrow M_{jk} \partial u_0 \partial x_k \, d\mu \)

\[
\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j \, dx = \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j \, dx
\]
To prove \[ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \partial u_\epsilon \frac{\partial x_j}{\partial x_j} dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} d\mu \]

\[ \frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j dx = \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j dx \]

\[ \parallel \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx \parallel \]
To prove \( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \to \sum_{jk} M_{jk} \frac{\partial u_0}{\partial x_k} d\mu \)

\[
\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j dx = \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j dx
\]

\[
\frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx = \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx
\]
To prove \[ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \, dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} d\mu \]

\[ \frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon \, dx = \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0 \, dx \]

\[ \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon \, dx \]

\[ \int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \]
To prove \( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \, dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \)

\[
\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j \, dx \quad = \quad \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j \, dx
\]

\[
\frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j \, dx
\]

\[
\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu
\]

\[
\int (\gamma_0 - \gamma_1) \lim_{\epsilon \rightarrow 0} \left( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \right) \, dx
\]
To prove \( \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_j} \, dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \)

\[
\frac{1}{|\omega_\varepsilon|} \int \gamma_\varepsilon \nabla (u_\varepsilon - u_0) \nabla v^j_\varepsilon \, dx = \frac{1}{|\omega_\varepsilon|} \int \gamma_0 \nabla (u_\varepsilon - u_0) \nabla v^j_0 \, dx
\]

\[
\frac{1}{|\omega_\varepsilon|} \int (\gamma_0 - \gamma_\varepsilon) \nabla u_0 \nabla v^j_\varepsilon \, dx \Downarrow \frac{1}{|\omega_\varepsilon|} \int (\gamma_0 - \gamma_\varepsilon) \nabla u_\varepsilon \nabla v^j_0 \, dx
\]

\[
\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \Downarrow \int (\gamma_0 - \gamma_1) \lim_{\varepsilon \to 0} \left( \frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_j} \right) \, dx
\]

+ compensated compactness (judicious integration by parts)
To prove \[ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \, dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \]

\[ \frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v^j_\epsilon \, dx = \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v^j_0 \, dx \]

\[ \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v^j_\epsilon \, dx \]

\[ \int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} \, d\mu \]

\[ \int (\gamma_0 - \gamma_1) \lim_{\epsilon \rightarrow 0} \left( \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \right) \, dx \]

+ compensated compactness (judicious integration by parts)

+ cut-offs.
Bounds and variational characterization of $M$

$$
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j d\mu = \frac{1}{|\omega_\epsilon|} \min_{w \in H^1(\Omega)} \int_{\Omega} \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 dx \\
+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 dx + o(1) .
$$
Bounds and variational characterization of $M$

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu = \frac{1}{|\omega_\epsilon|} \min_{w \in H^1(\Omega)} \int_{\Omega} \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi dx \\
+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx + o(1)
\]

for all uniformly positive, smooth functions $\phi$ on $\Omega$. 
Bounds and variational characterization of $M$

$$\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu = \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{per}(Q)} \int_{Q} \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi dx$$

$$+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx + o(1) .$$

for all uniformly positive, smooth functions $\phi$ on $Q$, $\Omega \subset Q$. 
Bounds and variational characterization of $M$

\[
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu = \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{per}(Q)} \int_{Q} \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} \omega_\epsilon \xi \right|^2 \phi dx \\
+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx + o(1) .
\]

for all uniformly positive, smooth functions $\phi$ on $Q$, $\Omega \subset Q$.

This yields

\[
\min \left(1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 \leq M(x) \xi \cdot \xi \leq \max \left(1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 ,
\]

for $\xi \in IR^n$, $\mu$ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$. 
Bounds and variational characterization of $M$

$$
\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu = \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{\text{per}}(Q)} \int_Q \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi dx
$$

$$
+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx + o(1) .
$$

for all uniformly positive, smooth functions $\phi$ on $Q$, $\Omega \subset Q$.

This yields

$$
\min \left( 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 \leq M(x) \xi \cdot \xi \leq \max \left( 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 ,
$$

for $\xi \in \mathbb{R}^n$, $\mu$ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$.

It also yields
Bounds and variational characterization of $M$

\[
\text{Trace } (M(x)) \leq n - 1 + \frac{\gamma_0(x)}{\gamma_1(x)}
\]

\[
\text{Trace } (M^{-1}(x)) \leq n - 1 + \frac{\gamma_1(x)}{\gamma_0(x)}
\]

$\mu$ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$. 
Bounds and variational characterization of $M$

\[
\begin{align*}
\text{Trace} \left( M(x) \right) & \leq n - 1 + \frac{\gamma_0(x)}{\gamma_1(x)} \\
\text{Trace} \left( M^{-1}(x) \right) & \leq n - 1 + \frac{\gamma_1(x)}{\gamma_0(x)}
\end{align*}
\]

$\mu$ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$. 

\[\text{Graphical representation of bounds and variational characterization.}\]
Bounds

M satisfies

\[ \min \left(1, \frac{\gamma_0}{\gamma_1}\right) I_n \leq M \leq \max \left(1, \frac{\gamma_0}{\gamma_1}\right) I_n, \]

and its trace satisfies “tighter” bounds
Bounds

M satisfies

\[ \min \left( 1, \frac{\gamma_0}{\gamma_1} \right) I_n \leq M \leq \max \left( 1, \frac{\gamma_0}{\gamma_1} \right) I_n, \]

and its trace satisfies “tighter” bounds

\[ n h \left( 1, \ldots, 1, \frac{\gamma_0}{\gamma_1} \right) \leq \text{trace}(M) \leq n a \left( 1, \ldots, 1, \frac{\gamma_0}{\gamma_1} \right). \]

\( h \) is the harmonic average, and \( a \) is the arithmetic average.
Bounds

$M$ satisfies
\[ \min \left( 1, \frac{\gamma_0}{\gamma_1} \right) I_n \leq M \leq \max \left( 1, \frac{\gamma_0}{\gamma_1} \right) I_n, \]
and its trace satisfies “tighter” bounds
\[ n h \left( 1, \ldots, 1, \frac{\gamma_0}{\gamma_1} \right) \leq \text{trace}(M) \leq n a \left( 1, \ldots, 1, \frac{\gamma_0}{\gamma_1} \right). \]

$h$ is the harmonic average, and $a$ is the arithmetic average.

Three of these bounds are attained for a single “sheet-like” inclusion. In that case the polarization eigenvalues “parallel” to the sheet are $1$, and the eigenvalue across the sheet is $\frac{\gamma_0}{\gamma_1}$. The fourth bound is attained for a single inclusion in the shape of a ball.
∀ \ y ∈ \partial \Omega, \ u_\epsilon(y)−u_0(y) = |ω_\epsilon| \int_\Omega (γ_1−γ_0)M_{ij}(x) \frac{∂u_0}{∂x_j} \frac{∂N}{∂x_i}(x, y)dμ(x)+o(|ω_\epsilon|)

May be used:
∀y ∈ ∂Ω, \( u_ε(y) - u_0(y) = |ω_ε| \int_Ω (γ_1 - γ_0) M_{ij}(x) \frac{∂u_0}{∂x_j} \frac{∂N}{∂x_i}(x, y) dμ(x) + o(|ω_ε|) \)

May be used:

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).
∀ \ y \in \partial \Omega, \ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0)M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)

May be used:

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).

2. To estimate the volume of inhomogeneities of moderate size (Capdebooscq, MV)
Detecting locations

Suppose $\omega_\epsilon = \bigcup_{j=1}^p (z_j + \epsilon B_j)$ (the inhomogeneities “shrink” to points $z_j$).

$$D(\phi)(\cdot) = |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, \cdot) d\mu(x)$$

$$= |\omega_\epsilon| \sum_{j=1}^p (\gamma_1 - \gamma_0) \alpha_j M^j \nabla u_0(z_j) \cdot \nabla_x N(z_j, \cdot)$$

Is linear in $\phi$ (the prescribed boundary condition), its range is finite dimensional (of dimension $np$).
Detecting locations

Suppose \( \omega_\epsilon = \bigcup_{j=1}^{p} (z_j + \epsilon B_j) \) (the inhomogeneities “shrink” to points \( z_j \)).

\[
D(\phi)(\cdot) = |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i} (x, \cdot) d\mu(x)
\]

\[
= |\omega_\epsilon| \sum_{j=1}^{p} (\gamma_1 - \gamma_0) \alpha_j M^j \nabla u_0(z_j) \cdot \nabla_x N(z_j, \cdot)
\]

Is linear in \( \phi \) (the prescribed boundary condition), its range is finite dimensional (of dimension \( np \)). In fact,

\[
\mathcal{R}(D) = \text{span}\{ e_k \cdot \nabla_x N(z_j, \cdot) | \partial B : k = 1, \ldots, n, j = 1, \ldots, p \}.
\]

Probe with \( g_{z,d} = d \cdot \nabla_x N(z, \cdot) | \partial B \). Then \( g_{z,d} \in \mathcal{R}(D) \) iff \( z \in \{ z_j : j = 1, \ldots, p \} \). Note also that \( \mathcal{R}(D) \) is well approximated by \( \mathcal{R}(\Lambda_\epsilon - \Lambda_0) \) (the measured Neumann-Dirichlet “difference” map).
Detecting locations

Method:
Detecting locations

Method:

1. Compute the SVD decomposition of $\Lambda_\epsilon - \Lambda_0$, and the projector onto the space spanned by the first $m$ eigenvectors, $P_m$. 
Detecting locations

Method:

1. Compute the SVD decomposition of $\Lambda_\epsilon - \Lambda_0$, and the projector onto the space spanned by the first $m$ eigenvectors, $P_m$.

2. For a test point $z$, compute

$$\cot \theta_m(z) = \frac{\|P_m g_{z,d}\|}{\|(I - P_m) g_{z,d}\|}.$$ 

For $m = pn$, $z \in \{z_j : j = 1, \ldots, p\} \Leftrightarrow \cot \theta_m(z) = \infty$. 
Detecting locations
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Detecting locations
Volume estimation

Suppose $\gamma_0$ and $\gamma_1$ are constants.

\[
u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x)\]
Volume estimation

Suppose $\gamma_0$ and $\gamma_1$ are constants.

\[
 u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i} (x, y) d\mu(x)
\]

\[
 \int_{\Omega} (u_\epsilon - u_0) \phi dy = -|\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} d\mu(x)
\]
Volume estimation

Suppose $\gamma_0$ and $\gamma_1$ are constants.

$$u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x)$$

$$\int_\Omega (u_\epsilon - u_0) \phi dy = -|\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} d\mu(x)$$

measured data
Volume estimation

Suppose \( \gamma_0 \) and \( \gamma_1 \) are constants.

\[
\begin{align*}
u_\varepsilon(y) - u_0(y) &= |\omega_\varepsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) \\
\int_\Omega (u_\varepsilon - u_0) \phi dy &= -|\omega_\varepsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} d\mu(x)
\end{align*}
\]

measured data

Pick \( \nabla u_0 = e_j, \ j = 1, \ldots, n \)
Volume estimation

Suppose $\gamma_0$ and $\gamma_1$ are constants.

\[ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) \]

\[ \int_\Omega (u_\epsilon - u_0) \phi dy = -|\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_i} d\mu(x) \]

measured data

Pick $\nabla u_0 = e_j, j = 1, \ldots, n$

\[ \frac{\text{data}_j}{\gamma_1 - \gamma_0} = |\omega_\epsilon| \int_\Omega M_{jj} d\mu, \text{ and } \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \text{trace} \left( \int_\Omega M d\mu \right). \]
Volume estimation

Using the bounds on $M$ we obtain

\[
\min \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq \max \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right|.
\]
Volume estimation

Using the bounds on M we obtain

\[
\min \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq \max \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right|.
\]

One measurement bounds. Valid in general (Alessandrini, Rosset, Seo).
Volume estimation

Using the bounds on $M$ we obtain

$$\min \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq \max \left( 1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right|.$$  

One measurement bounds. Valid in general (Alessandrini, Rosset, Seo).

$$h \left( \frac{1, \ldots, 1, \frac{\gamma_1}{\gamma_0}}{n} \right) \left| \frac{\sum_{j=1}^{n} \text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq a \left( \frac{1, \ldots, 1, \frac{\gamma_1}{\gamma_0}}{n} \right) \left| \frac{\sum_{j=1}^{n} \text{data}_j}{\gamma_1 - \gamma_0} \right|$$

$n$ measurement bounds. Only asymptotic bounds.
Random Ellipses
Volume estimation, with $\gamma_0 > \gamma_1$

Proportion of total volume occupied by the inhomogeneities.
Volume estimation, with $\gamma_0 > \gamma_1$

Proportion of total volume occupied by the inhomogeneities.

$A$: the upper estimate obtained from one measurement bounds.
Volume estimation, with $\gamma_0 > \gamma_1$

Proportion of total volume occupied by the inhomogeneities.

$A$: the upper estimate obtained from one measurement bounds.

$B$: the lower estimate obtained from one measurement bounds.
Volume estimation, with $\gamma_0 > \gamma_1$

Proportion of total volume occupied by the inhomogeneities.

* $A$: the upper estimate obtained from one measurement bounds.
* $B$: the lower estimate obtained from one measurement bounds.
* $C$: the upper estimate obtained from two measurement bounds.
Volume estimation, with $\gamma_0 > \gamma_1$

Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

C: the upper estimate obtained from two measurement bounds.

D: the lower estimate obtained from two measurement bounds.
Volume estimation, with $\gamma_1 > \gamma_0$

Proportion of total volume occupied by the inhomogeneities.
Volume estimation, with $\gamma_1 > \gamma_0$

Proportion of total volume occupied by the inhomogeneities.

$A$: the upper estimate obtained from one measurement bounds.
Volume estimation, with $\gamma_1 > \gamma_0$

Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.
Volume estimation, with $\gamma_1 > \gamma_0$

Proportion of total volume occupied by the inhomogeneities.

**A**: the upper estimate obtained from one measurement bounds.

**B**: the lower estimate obtained from one measurement bounds.

**C**: the upper estimate obtained from two measurement bounds.
Volume estimation, with $\gamma_1 > \gamma_0$

Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.
B: the lower estimate obtained from one measurement bounds.
C: the upper estimate obtained from two measurement bounds.
D: the lower estimate obtained from two measurement bounds.
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

\[ \nabla \cdot \left( \frac{1}{\mu_\varepsilon} \nabla E_\varepsilon \right) + k^2 q_\varepsilon E_\varepsilon = 0 , \]  
with
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

\[ \nabla \cdot \left( \frac{1}{\mu_\epsilon} \nabla E_\epsilon \right) + k^2 q_\epsilon E_\epsilon = 0 \ , \ \text{with} \ \ \omega_\epsilon = \bigcup_{j=1}^{p} (z_j + \epsilon B_j) \subset \mathbb{R}^2. \]
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

\[ \nabla \cdot \left( \frac{1}{\mu_\varepsilon} \nabla E_\varepsilon \right) + k^2 q_\varepsilon E_\varepsilon = 0 \ , \ \text{with} \ \ \omega_\varepsilon = \bigcup_{j=1}^p (z_j + \varepsilon B_j) \subset \mathbb{R}^2 . \]

In that case, for fixed \( k > 0 \),

\[
E_\varepsilon(y) - E_0(y) = \varepsilon^2 \sum_{j=1}^p \left( \frac{\mu_0}{\mu_j} - 1 \right) |B_j| M_j \left( \frac{\mu_j}{\mu_0} \right) \nabla E_0(z_j) \cdot \nabla_x \Phi^k(z_j, y) \]

\[
+ (\varepsilon k)^2 \mu_0 q_0 \sum_{j=1}^p \left( 1 - \frac{q_j}{q_0} \right) |B_j| E_0(z_j) \Phi^k(z_j, y) + o(\varepsilon^2) .
\]
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

\[ \nabla \cdot \left( \frac{1}{\mu_\varepsilon} \nabla E_\varepsilon \right) + k^2 q_\varepsilon E_\varepsilon = 0 \ , \ \text{with} \ \omega_\varepsilon = \bigcup_{j=1}^{p} (z_j + \varepsilon B_j) \subset \mathbb{R}^2 \ . \]

In that case, for \textit{fixed} \ \( k > 0 \),

\[
E_\varepsilon(y) - E_0(y) = \varepsilon^2 \sum_{j=1}^{p} \left( \frac{\mu_0}{\mu_j} - 1 \right) |B_j| M^j \left( \frac{\mu_j}{\mu_0} \right) \nabla E_0(z_j) \cdot \nabla_x \Phi^k(z_j, y) \\
+ (\varepsilon k)^2 \mu_0 q_0 \sum_{j=1}^{p} (1 - \frac{q_j}{q_0}) |B_j| E_0(z_j) \Phi^k(z_j, y) + o(\varepsilon^2)
\]

With \( \Phi^k \) solving \((\Delta + k^2 \mu_0 q_0) \Phi^k(\cdot, y) = \delta_y\)
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

\[ \nabla \cdot \left( \frac{1}{\mu_\epsilon} \nabla E_\epsilon \right) + k^2 q_\epsilon E_\epsilon = 0 \ , \quad \text{with} \quad \omega_\epsilon = \bigcup_{j=1}^{p} (z_j + \epsilon B_j) \subset \mathbb{R}^2 \ . \]

In that case, for fixed \( k > 0 \),

\[
E_\epsilon(y) - E_0(y) = \epsilon^2 \sum_{j=1}^{p} \left( \frac{\mu_0}{\mu_j} - 1 \right) |B_j| M_j \left( \frac{\mu_j}{\mu_0} \right) \nabla E_0(z_j) \cdot \nabla_x \Phi^k(z_j, y) \\
+ (\epsilon k)^2 \mu_0 q_0 \sum_{j=1}^{p} \left( 1 - \frac{q_j}{q_0} \right) |B_j| E_0(z_j) \Phi^k(z_j, y) + o(\epsilon^2)
\]

With \( \Phi^k \) solving \( (\Delta + k^2 \mu_0 q_0) \Phi^k(\cdot, y) = \delta_y \)

We now take \( E_0 \) to be a plane wave, \( i.e., \)

\[
\Phi^k(x, y) = -\frac{i}{4} H_0^{(1)}(k \sqrt{\mu_0 q_0} |x - y|) \quad \text{and} \quad E_0(x) = e^{ik \sqrt{\mu_0 q_0} \xi \cdot x}
\]

The first two terms are now of order \( (\epsilon k)^2 / (1 + \sqrt{k}) \) and it is possible to prove that the remainder is of order \( o((\epsilon k)^2) / (1 + \sqrt{k}) \), as \( \epsilon k \to 0 \), \( k \geq k_0 > 0 \).
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

In other words, “our small inhomogeneity approximation” holds uniformly as $\epsilon k \to 0$, and not just for fixed $k$ as $\epsilon \to 0$. 
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As $\epsilon k \to \lambda_0 > 0$ we have $k \approx \epsilon^{-1}$, and so $(\epsilon k)^2/(1 + \sqrt{k}) \approx \sqrt{\epsilon}$, i.e., we expect $E_\epsilon(y) - E_0(y) = O(\sqrt{\epsilon})$. 

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We now show the $L^2$ norm of $E_\epsilon(y) - E_0(y)$ on the circle $|y| = 2$ for three circular inhomogeneities of radius $\epsilon = 0.01$, $4 \cdot 0.001$, and $0.001$, respectively.
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)
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These were for a conducting inhomogeneity ($\mu_1 = 2$, $q_1 = 2 + 2i$ inside the inhomogeneity, $\mu_0 = q_0 = 1$ outside). For a non-conducting inhomogeneity a similar computation gives
The 2d Helmholtz Equation (Transverse Magnetic Maxwell)
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One inhomogeneity of the form $\epsilon D$; change variables $x \rightarrow z = \frac{x}{\epsilon}$, $k \rightarrow \lambda = k\epsilon$, $V_\lambda(z) = E_{k,\epsilon}(z\epsilon)$. Then

$$\nabla_z \cdot \left( \frac{1}{\mu} \nabla_z V \right) + \lambda^2 qV = 0 \quad \text{in } \mathbb{R}^2.$$

Introduce $V_\lambda^{(s)}(z) = V_\lambda(z) - e^{i\lambda\sqrt{\mu_0 q_0} \xi \cdot z}$, then

$$E_{k,\epsilon}(y) - E_{k,0}(y) = V_\lambda^{(s)}\left(\frac{y}{\epsilon}\right)$$

$$= \int_{\partial D} \frac{\partial}{\partial n_x} \Phi^\lambda(x, \frac{y}{\epsilon}) V_\lambda^{(s)}(x) \, d\sigma_x$$

$$- \int_{\partial D} \Phi^\lambda(x, \frac{y}{\epsilon}) \frac{\partial}{\partial n} V_\lambda^{(s)}(x) \, d\sigma_x$$
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(1) $\lambda \to 0$ gives the low frequency formulas from before,

(2) $\lambda \to \lambda_0 > 0$ gives an intermediate limit formula
    (by continuity),

(3) $\lambda \to \infty$ may, e.g., be treated by a combination with
    appropriate “geometric optics” and stationary phase.
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