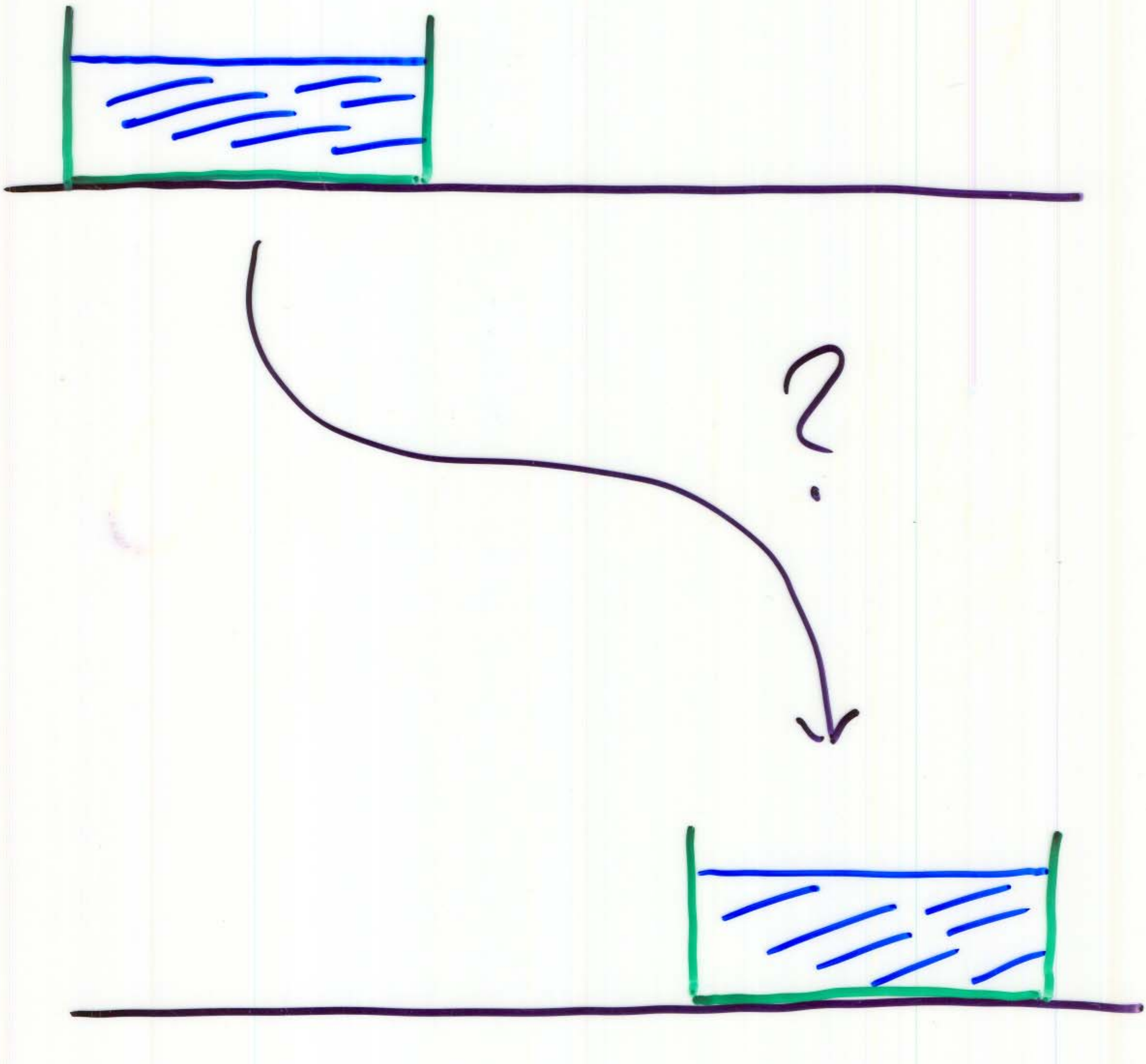
Contrôlabilité et non linéarité

Controllability and nonlinearity

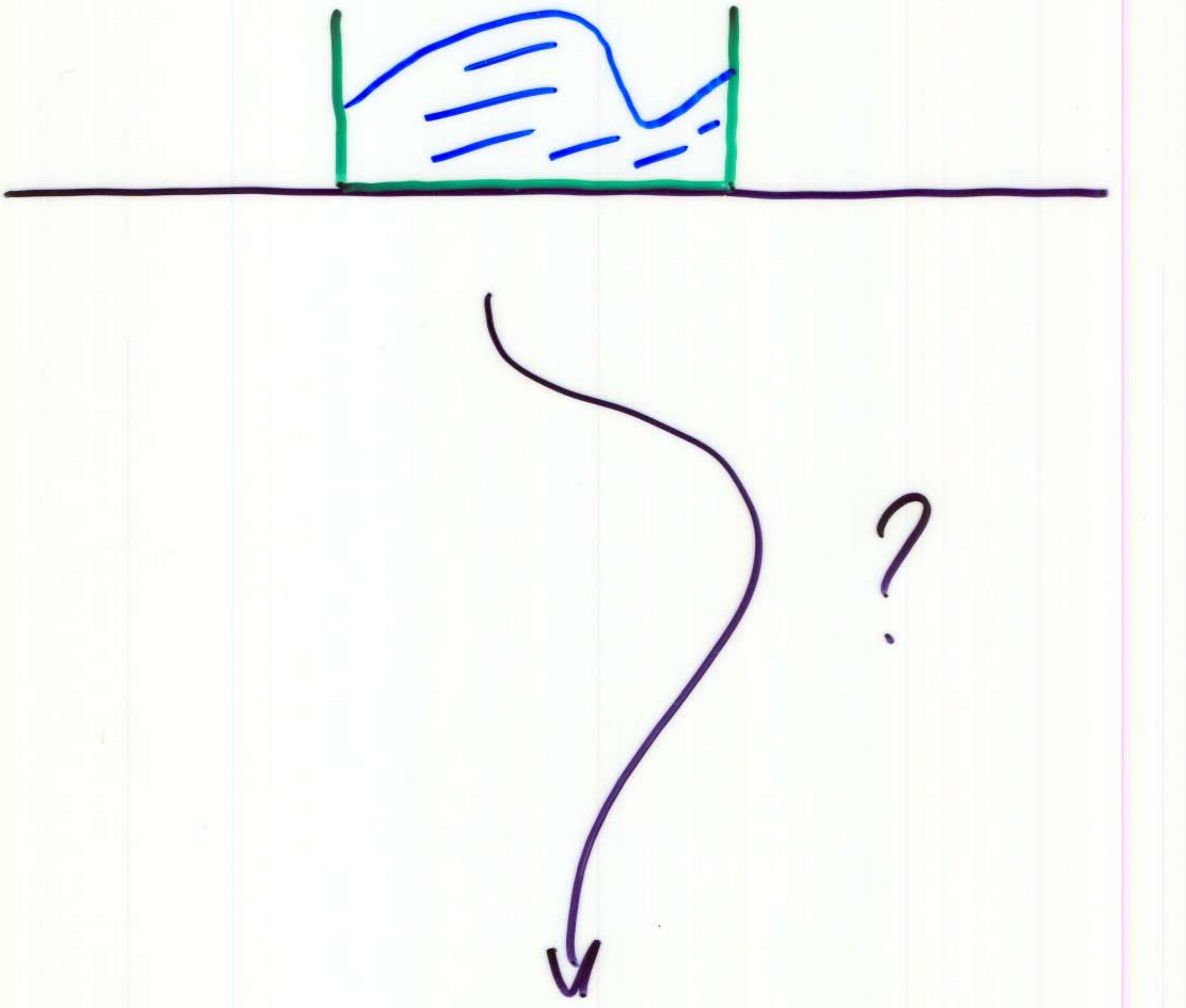
CANUM 2006

Pb 1



Pb raised by P. Rouchon

Pb 2

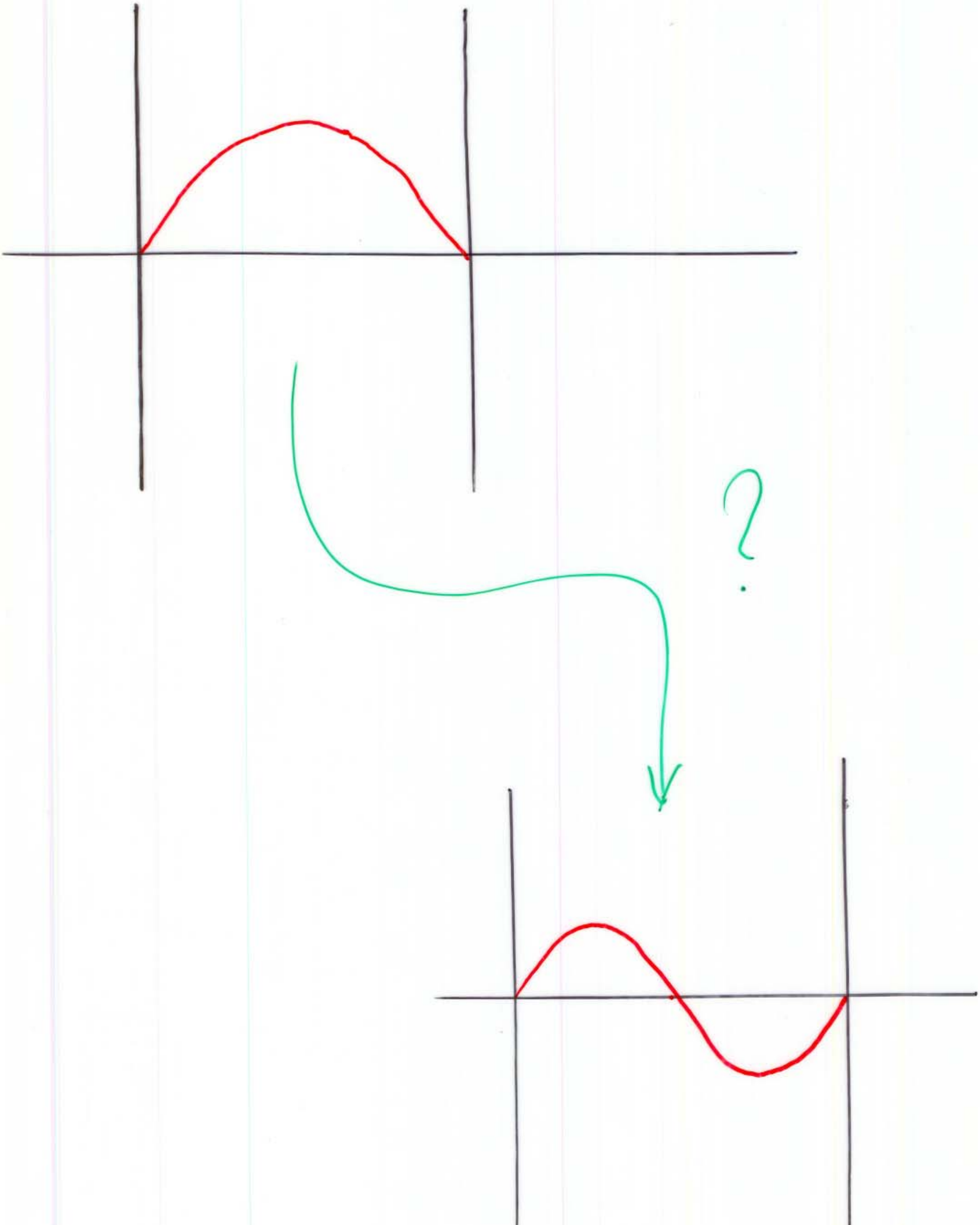


Yes

- Local result

- Saint-Venant equations (shallow water equations) (C. 2002)

Schrödinger



Yes

K. Beauchard (2005)

K. Beauchard
\+ J.-M. C } (2005)

T has to be large enough

JMC 2006

$$\dot{x} = f(\overset{\in \mathbb{R}^n}{x}, \underset{\in \mathbb{R}^m}{u})$$

$$f(0,0) = 0$$

Pb

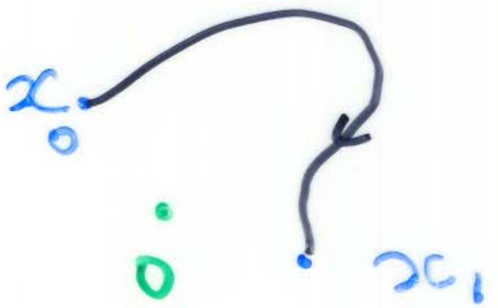
(local) controllability:

$\forall x_0 \in \mathbb{R}^n$, $\forall x_1 \in \mathbb{R}^n$

(with $|x_0| + |x_1|$ small)

$\exists u : [0,T] \to \mathbb{R}^m$

$$\left.\begin{aligned} \dot{x} &= f(x, u(t)) \\ x(0) &= x_0 \end{aligned}\right\} \Rightarrow x(T) = x_1$$



linearize around (0,0)

$$\dot{x} = \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial u}(0,0)\,u$$

Controllable

$\Downarrow$ ← inverse mapping theorem

$$\dot{x} = f(x, u)$$

is locally controllable

Example

(Car model)
Nonholonomic integrator
Brockett's system

$$\begin{rcases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1 \end{rcases} \circledast$$

linearize around $(x=0, u=0)$

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = 0$$

$\to$ not controllable

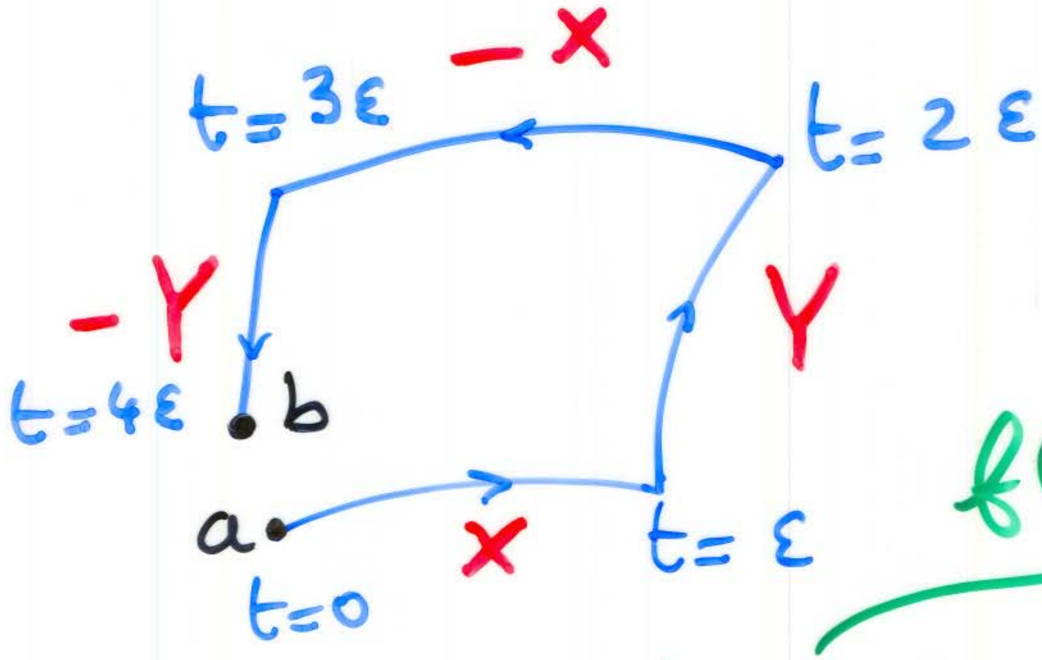
But $\circledast$ is controllable

For nonlinear systems

- No necessary and sufficient condition known
- Many necessary conditions, many sufficient conditions known

Agrachev
Bianchini
Frankowska
Gamkrelidze
Hermes
Kawski
Krener
Stefani
Sussmann

Usual tool
Lie Bracket



$$\dot{x} = \overbrace{u_1 X + u_2 Y}^{f(x,u)}$$

$$b = a + \varepsilon^2 [X, Y](a) + O(\varepsilon^3) \quad \varepsilon \to 0$$

$$[X, Y]^i = \sum_{j=1}^{n} X^j \frac{\partial Y^i}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j}$$

$$\dot{x} = f_0(x) + u f_1(x)$$

$$f_0(a) = 0$$

$$u(t) = \eta \qquad t \in (0, \varepsilon)$$

$$u(t) = -\eta \qquad t \in (\varepsilon, 2\varepsilon)$$

$t = 2\varepsilon$

$$b = a - \varepsilon^2 \eta [f_0, f_1](a) + O(\varepsilon^3)$$

as $\varepsilon \to 0^+$

Example

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\dot{x}_3 = x_1 u_2 - x_2 u_1$$

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$$

$$[f_1, f_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Span $= \mathbb{R}^3$

$\Downarrow$

Controllability

Hermes condition

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \quad f_0(0) = 0$$

Br : set of iterated Lie brackets of $f_0, \cdots, f_m$

Ex : $h := [f_1, [f_1, f_0]] \in Br$

$h \in Br$ $\delta_i(h)$: number of f_i

$\delta_0(h) = 1$, $\delta_1(h) = 2$ $\delta_i(h) = 0$, $i \in \{2, \cdots, m\}$

Th H_1 : Span$\{h(0); h \in Br\} = \mathbb{R}^n$

H_2 : $h \in Br$ $\delta_0(h)$ odd, $\delta_i(h)$ even for $i \in \{1, \dots, m\}$

$\Rightarrow h(0) \in$ Span$\{g(0) \in Br;$

$\sum_{i=1}^{m} \delta_i(g) < \sum_{i=1}^{m} \delta_i(h)\}$

$\Rightarrow$ local controllability (in small time)

Hermes conjecture plane 1976
special cases 1982

Sussmann 1983 (more general results 1987)

Agrachev - Sarychev 2005

Shirikyan 2006

(Navier-Stokes + Euler)

Ledyaev

$$\dot{x} = \sum_{i=1}^{\infty} u_i f_i(x)$$

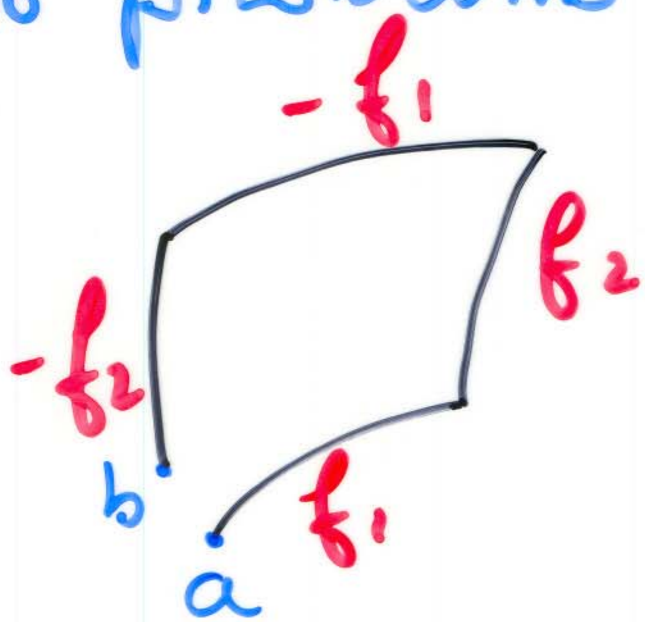
2004

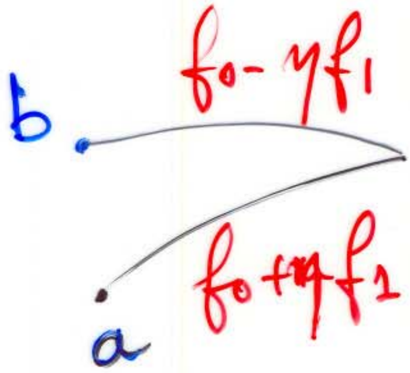
Infinite dimension

Lie brackets

Two problems

(1)





Usually $\frac{b-a}{\varepsilon^2} \not\to$ as $\varepsilon \to 0^+$

(2) Estimates

$$y^\varepsilon_t + y^\varepsilon_x = 0 \qquad 0<x<L$$

$$y^\varepsilon(t,0) = u^\varepsilon(t)$$

$$y^\varepsilon(0,x) = 0$$

$$u^\varepsilon(t) = \begin{cases} 1 & 0<t<\varepsilon \\ -1 & \varepsilon<t<2\varepsilon \end{cases}$$

$$y^\varepsilon(2\varepsilon,x) = \begin{cases} -1 & 0<x<\varepsilon \\ 1 & \varepsilon<x<2\varepsilon<L \\ 0 & x>2\varepsilon \end{cases}$$

$$\frac{y^\varepsilon(2\varepsilon,\cdot)-y^\varepsilon(0,\cdot)}{\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} 0 \underbrace{+\,\delta_0'}$$

What to do with this term

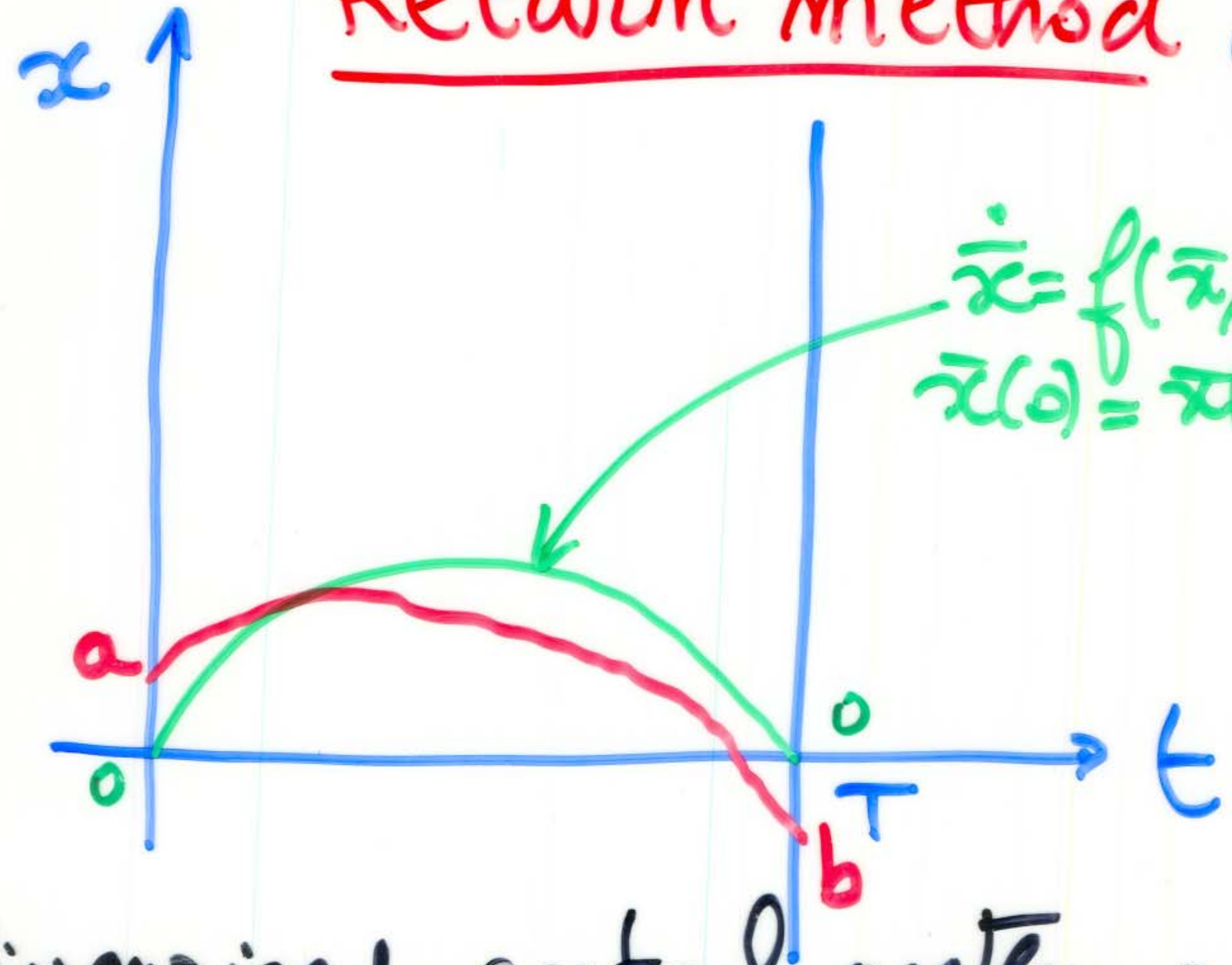
In fact $\dfrac{y^\varepsilon(\varepsilon,\cdot)-y^\varepsilon(0,\cdot)}{\varepsilon} \longrightarrow \delta_0$

Three methods

- Return method
- Quasi-static deformations
- Power series expansion

Return method

(C. 92)
(C. 98)



$\dot{\bar{x}} = f(\bar{x}, \bar{u}(t))$

$\bar{x}(0) = \bar{x}(T) = 0$

[linearized control system around $(\bar{x}, \bar{u})$ controllable]

$\Downarrow$ inverse mapping theorem

[local controllability of $\dot{x} = f(x,u)$]

$$\dot{\bar{x}}_1 = \bar{u}_1, \quad \dot{\bar{x}}_2 = \bar{u}_2, \quad \dot{\bar{x}}_3 = \bar{x}_1 \bar{u}_2 - \bar{x}_2 \bar{u}_1$$

$$\bar{x}(0) = (0,0,0)$$

$$\bar{u}(T-t) = -\bar{u}(t) \Longrightarrow \bar{x}(T-t) = \bar{x}(t)$$

$$\Downarrow$$

$$\bar{x}(T) = \bar{x}(0) = 0$$

linearized control system around $(\bar{x}, \bar{u})$:

$$\underbrace{\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{u}_2 & -\bar{u}_1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\bar{x}_2 & \bar{x}_1 \end{pmatrix} v}$$

Kalman's condition for time-varying linear systems

Controllable if (and only if)

$$\bar{u} \not\equiv 0$$

The return method allows to reduce the problem of the controllability of nonlinear systems to the controllability of linear systems

Applications to P.D.E.

Euler equations

C. 92-96, O. Glass 97-2000

Navier-Stokes equations

C. 96, A. Fursikov + O. Imanuvilov 99

Burgers equations

Th. Horsin 98

Shallow water equations

C. 2002

Vlasov-Poisson equations

O. Glass 2003

Schrödinger equations

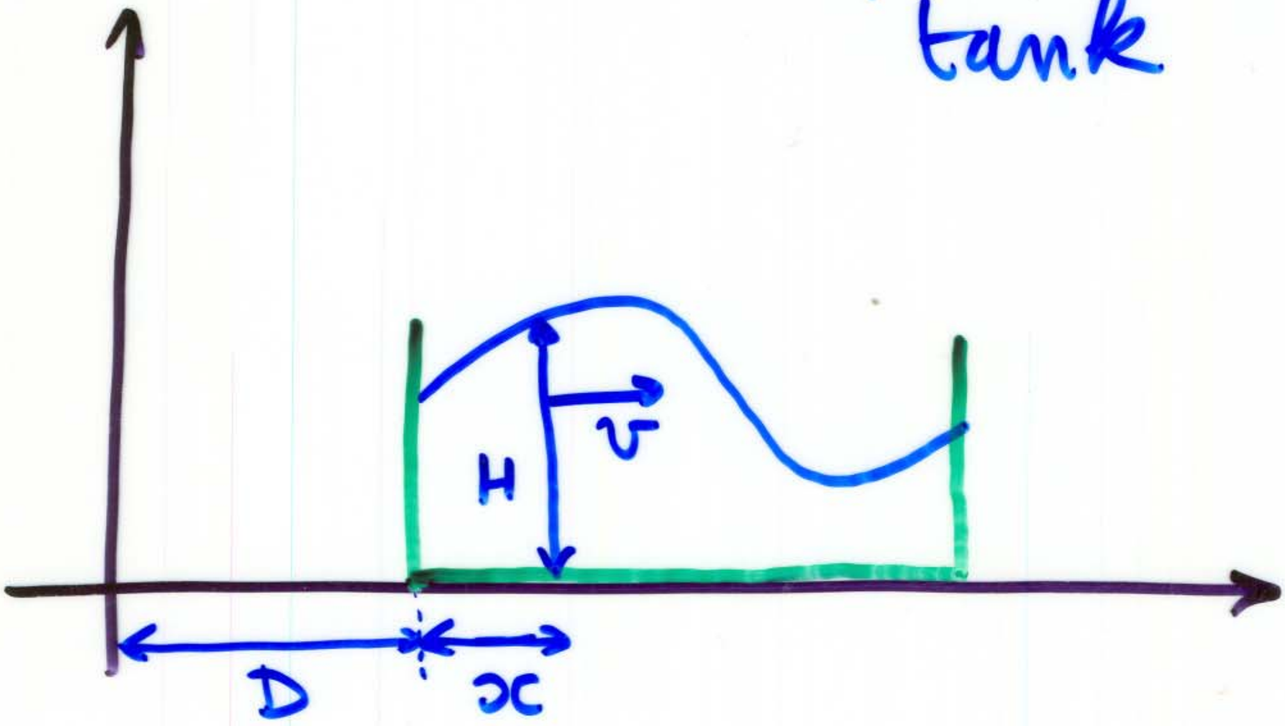
K. Beauchard 2005

K. " " + C. 2006

1-D isentropic Euler equations

O. Glass 2005

v : velocity with respect to the tank



$$H_t + (Hv)_x = 0$$

$$v_t + \left(gH + \frac{v^2}{2}\right)_x = -u(t)$$

acceleration of the tank

This is the control

$$\frac{ds}{dt} = u(t)$$

$$\frac{dD}{dt} = s$$

$$v(t,0) = v(t,L) = 0$$

State $Y = (H, v, s, D)$

$$\frac{d}{dt}\int_0^L H(t,x)\,dx = 0$$

$$\hookrightarrow \int_0^L H(t,x)\,dx = L.H_e$$

$$H_x(t,0) = H_x(t,L) \quad \left(= -\frac{u(t)}{g}\right)$$

State space $\mathcal{Y} = \{ (H, v, S, D)$ s.t.

$H \in C^1([0,L])$,

$\int_0^L H\,dx = L\,H_e$, $H_x(0) = H_x(L)$

$v \in C^1([0,L])$

$v(0) = v(L) = 0$

$S \in \mathbb{R}$

$D \in \mathbb{R}\}$

Th (C. 2002) $\exists T > 0 \; \exists C > 0 \; \exists \varepsilon > 0$ s.t.

$\forall (H_0, v_0, S_0, D_0) \in \mathcal{Y}$

$\forall (H_1, v_1, S_1, D_1) \in \mathcal{Y}$ with

$|H_0 - H_e|_{C^1} + |v_0|_{C^1} + |H_1 - H_e|_{C^1} + |v_1|_{C^1}$

$+ |S_1 - S_0| + |D_1 - S_0 T - D_0| \leq \varepsilon$

$\exists (H, v) \in C^1([0,T]\times[0,L])^2$

$\exists (S, D, u) \in C^1([0,T])^2 \times C^0([0,T])$ s.t.

$$H_t + (Hv)_x = 0$$

$$v_t + \left(gH + \frac{v^2}{2}\right)_x = -u(t)$$

$$v(t,0) = v(t,L) = 0$$

$$\dot{S} = u, \quad \dot{D} = S$$

$H(0,x) = H_0(x)$, $H(T,x) = H_1(x)$

$v(0,x) = v_0(x)$, $v(T,x) = v_1(x)$, $S(0) = S_0$

$S(T) = S_1$, $D(0) = D_0$, $D(T) = D_1$

$$|H(t,\cdot) - H_e|_{C^1} + |v(t,\cdot)|_{C^1} + |S(t)| + |D(t)| + |u(t)|$$

$$\leq C \Big\{ \left[|H_0 - H_e|_{C^1} + |H_1 - H_e|_{C^1} + |v_0|_{C^1} + |v_1|_{C^1}\right]^{1/2}$$

$$+ |S_1 - S_0| + |D_1 - S_0 T - D_0| \Big\}$$

Scaling $g = H_e = L = 1$

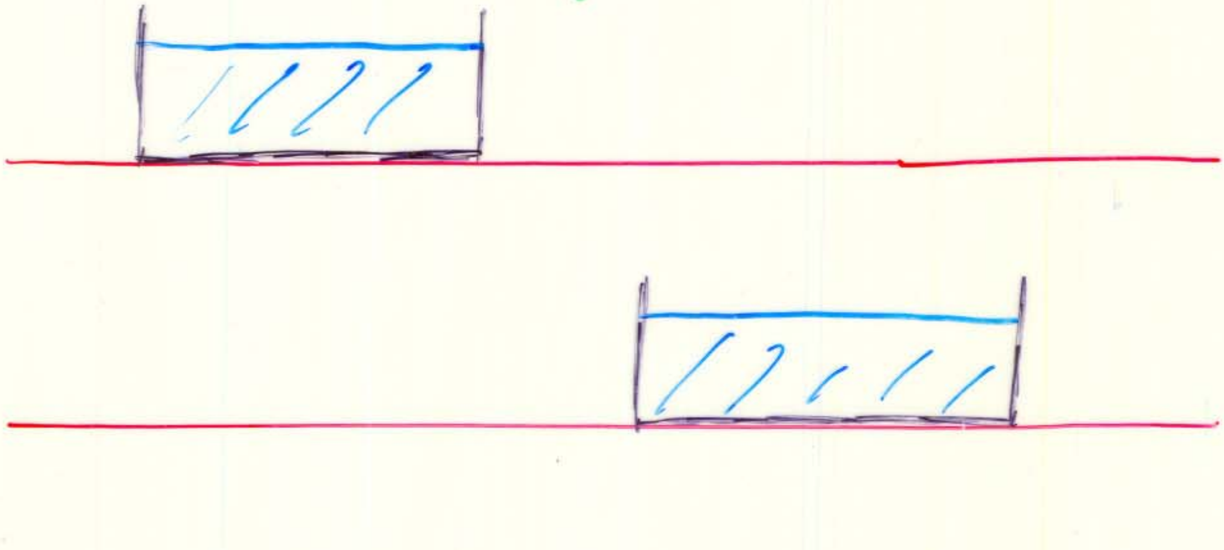
linearized control system around

$H = 1,\ v = 0,\ S = 0,\ D = 0$

$$(L)\quad \begin{cases} h_t + v_x = 0 \\ v_t + h_x = -u(t) \\ v(t,0) = v(t,1) = 0 \\ \dot{S} = u \quad \dot{D} = S \end{cases}$$

Dubois Petit Rouchon (99)

(L) is not controllable, but for (L) one can solve



Remark

$$(N_1)\begin{cases}\dot{x}_1 = x_1 + u\\ \dot{x}_2 = x_1^2\\ \dot{S} = u\\ \dot{D} = S\end{cases} \qquad (N_2)\begin{cases}\dot{x}_1 = x_1 + u\\ \dot{x}_2 = x_1^3\\ \dot{S} = u\\ \dot{D} = S\end{cases}$$

linearization

(L)

$$\dot{x}_1 = x_1 + u,\ \dot{x}_2 = 0,\ \dot{s} = u,\ \dot{D} = S$$

For (L) one can move from $(0,0,0,0)$ to $(0,0,0,1)$

This is also the case for (N_2), but not for (N_1)

What about Saint-Venant?

Toy model

$$H = 1 + \varepsilon H_1 + \varepsilon^2 H_2$$
$$V = \varepsilon V_1 + \varepsilon^2 V_2$$
$$u = \varepsilon u_1$$

PDE part:

$$\text{①}\begin{cases} H_{1t} + V_{1x} = 0 \\ V_{1t} + H_{1x} = -u_1 \\ V_1(t,0) = V_1(t,1) = 0 \end{cases}$$

$$\text{②}\begin{cases} H_{2t} + V_{2x} = -(H_1 V_1)_x \\ V_{2t} + H_{2x} = -\left(\frac{V_1^2}{2}\right)_x \\ V_2(t,0) = V_2(t,1) = 0 \end{cases}$$

$$
\left.\begin{aligned}
&(1)\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \end{cases} \\
&(2)\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -x_3 + 2x_1x_2 \end{cases} \\
&\begin{cases} \dfrac{ds}{dt} = u \\ \dfrac{dD}{dt} = s \end{cases}
\end{aligned}\right\} (N)
$$

ODE part

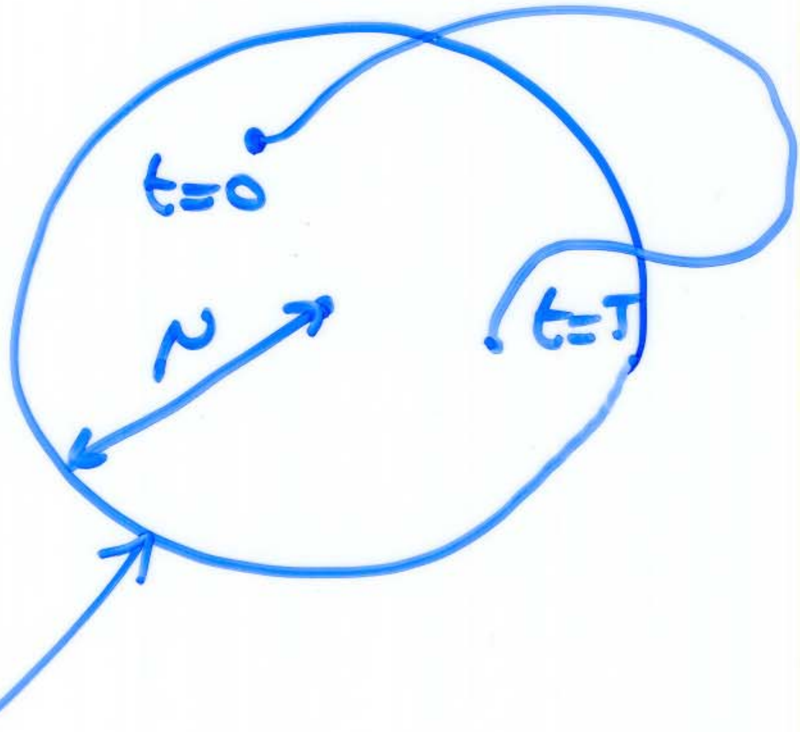
Linearization

$$
(L)\begin{cases} \dot{x}_1 = x_2, \ \dot{x}_2 = -x_1 + u, \ \dot{x}_3 = x_4 \\ \dot{x}_4 = -x_3, \ \dot{s} = u, \ \dot{D} = s \end{cases}
$$

For (L), $\forall \delta \in \mathbb{R}$ $\forall \varepsilon > 0$ one can move from $(0,0,0,0,0,0)$ to $(0,0,0,0,0,\delta)$ during the interval of time $[0, \varepsilon]$

But for (N) $\exists\, T > 0$

$\exists\, N > 0$



$$|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + S^2 + D^2 \leq N^2$$

$(T > \pi$ is OK$)$

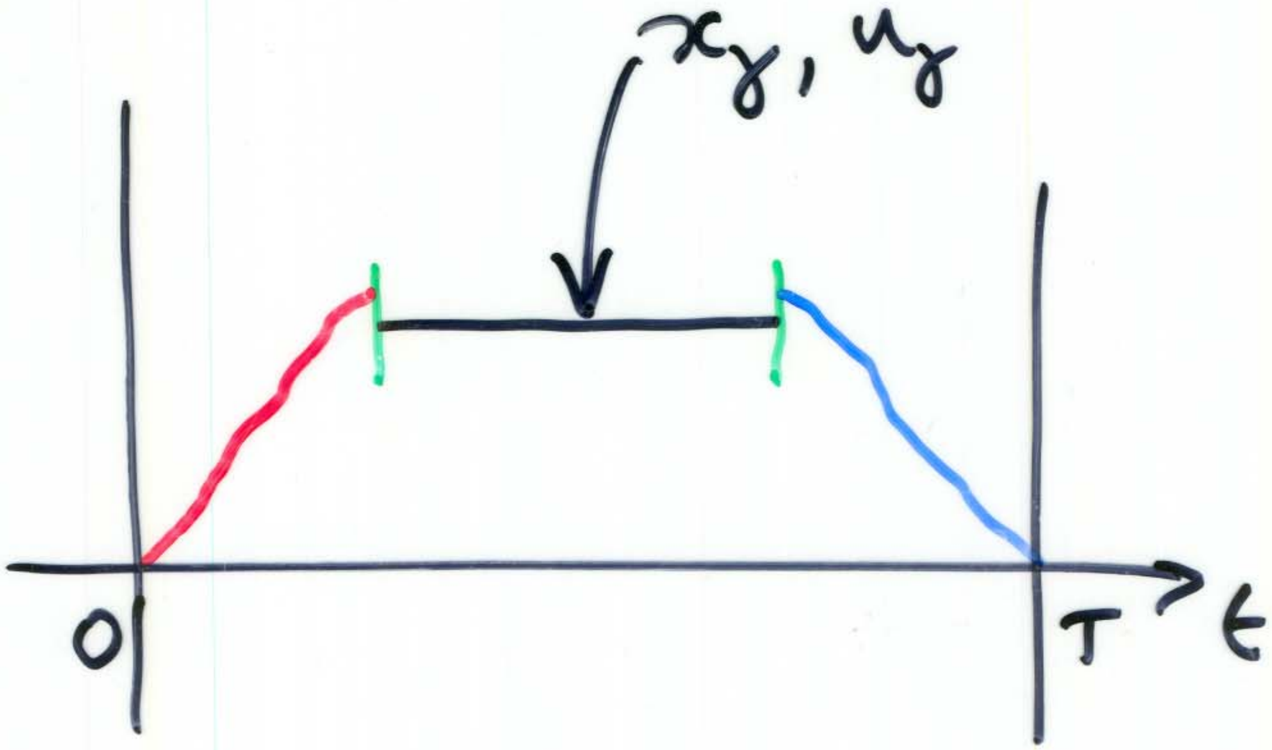
$\cancel{(S, D)}$

$\dot{x} = f(x, u)$

$$(B|AB|A^2B|A^3B) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 2\gamma & 0 \\ 0 & 2\gamma & 0 & -4\gamma \end{pmatrix}$$

$$\xrightarrow{\det} 12\gamma^2$$

(N) is locally controllable around (x_γ, u_γ)



How to construct the red trajectory (and the blue trajectory)

The linearized control system around $(H_\gamma, v_\gamma, S_\gamma, D_\gamma, \gamma)$ is

$$(L_\gamma)\begin{cases} h_t + \left(\left(1+\gamma\left(\frac{1}{2}-x\right)\right)v\right)_x = 0 \\ v_t + h_x = -u(t) \\ \dot{S} = u(t), \quad v(t,0) = v(t,1) = 0 \\ \dot{D} = S \end{cases}$$

(L_γ) is controllable for $\gamma \neq 0$ with $|\gamma|$ small

Iterative scheme

$$h_t^{n+1} + h^n v_x^{n+1} + v^n h_x^{n+1} = 0$$

$$v_t^{n+1} + h_x^{n+1} + v^n v_x^{n+1} = -u^{n+1}(t)$$

$$\dot{s}^{n+1} = u^{n+1} \qquad v^{n+1}(t,0) = v^{n+1}(t,1) = 0$$

$$\dot{D}^{n+1} = s^{n+1}$$

$(v^n, h^n) \in \Sigma$ of

codimension 4

$\longrightarrow$ controllability

insure $(v^{n+1}, h^{n+1}) \in \Sigma$

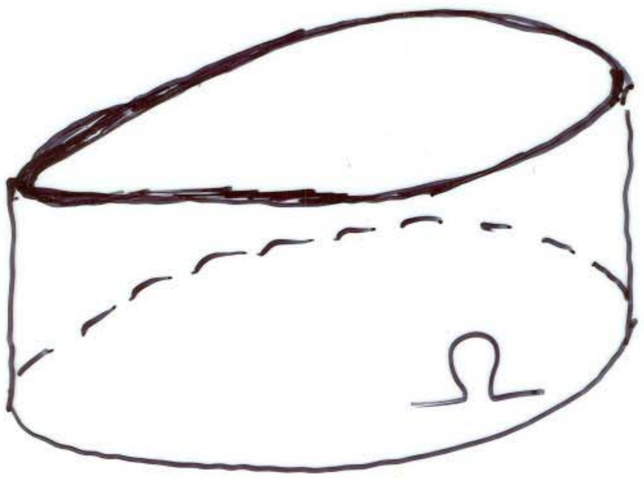
Open problem

Optimal value of T for local controllability

$T > 2\sqrt{\frac{H_e}{g}}$ enough ?

necessary ?

2-D Tank



Chitour - C. - Garavello (2004)

For generic Ω the linearized control system around $H = H_e$, $v = 0$, $S = 0$, $D = 0$ is <u>not</u> steady state controllable

Open pb
What about the <u>nonlinear</u> control system ?

Power series expansion

KdV

$$y_t + y_x + y_{xxx} + y y_x = 0$$

$$0 < x < L$$

$$y(t,0) = y(t,L) = 0$$

$$y_x(t,L) = u(t)$$

state $\quad y(t,\cdot)$

control $\quad u(t)$

Cauchy problem

locally well posed

$u(t) \in L^2(0,T)$

$y(0,x) \in L^2(0,L)$

$\rightarrow y \in C([0,T], L^2(0,L)) \cap L^2((0,T), H^1(0,L))$

Rosier 1997 linear case

C. + Crépeau 2003 nonlinear case

Local controllability

Rosier (1997)

local controllability

if

$$L \notin \left\{ 2\pi\sqrt{\frac{j^2+l^2+jl}{3}}\, ; j, l \in \mathbb{N}^* \right\}$$

Tool

- linearize
- HUM + observability inequality

$j = \ell = 1 \longrightarrow L = 2\pi$

Linearization $\rightarrow$

$$\left.\begin{array}{l} y_t + y_x + y_{xxx} = 0 \\ y(t,0) = y(t,2\pi) = 0 \\ y_x(t,2\pi) = u(t) \leftarrow \text{Control} \end{array}\right\} (\mathcal{L})$$

$$\rightarrow \frac{d}{dt}\int_0^{2\pi} y(1-\cos x)\,dx = 0$$

$(\mathcal{L})$ is not controllable

Rosier (1997)

Controllability of $(\mathcal{L})$ in
$\left\{ y \in L^2(0,2\pi); \int_0^{2\pi} y(1-\cos x)\,dx = 0 \right\}$

Try to move in the directions $\pm(1-\cos x)$

C + Crépeau 2003

local controllability for $L = 2\pi$ in small time

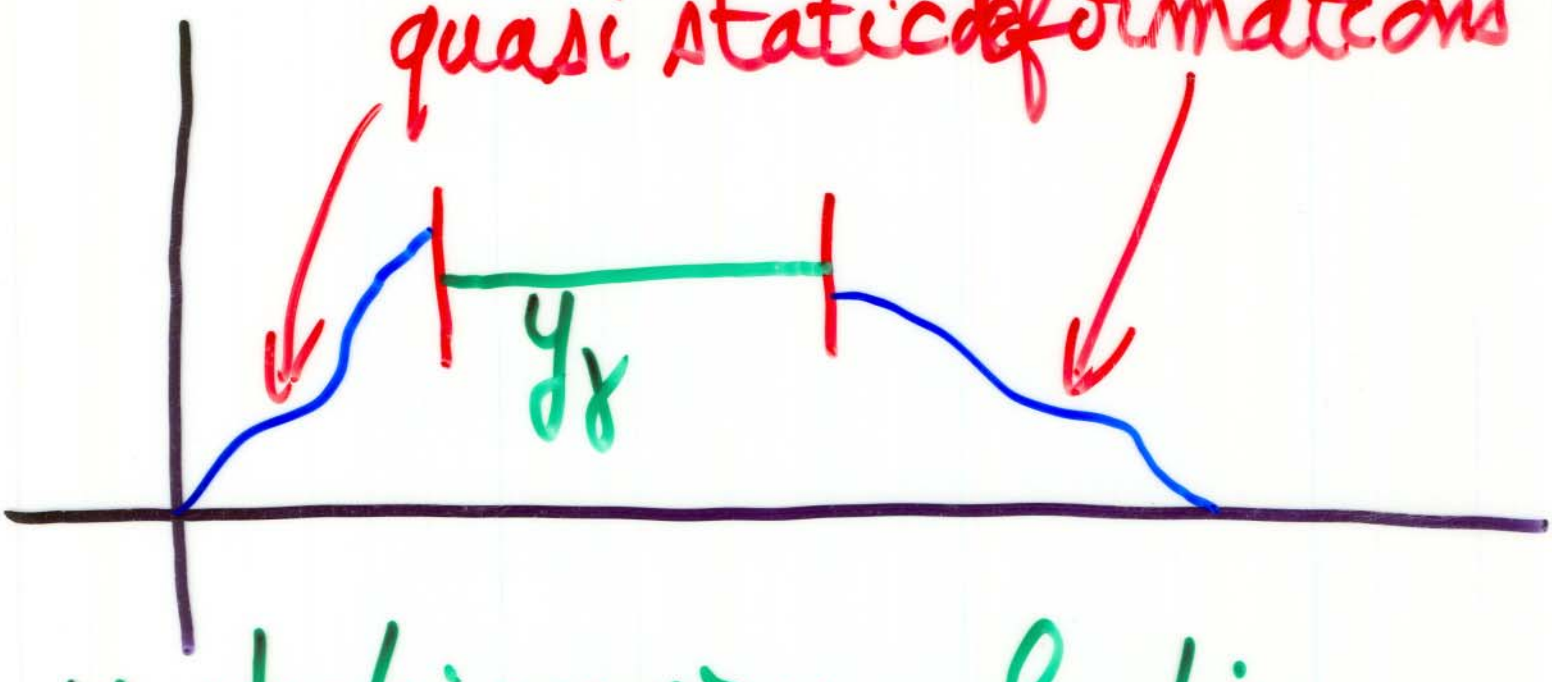
Method:

power series expansion

Remark

Return method + quasi static deformation

$\rightarrow$ gives local controllability in <u>large</u> time



y_γ stationary solution $\not\equiv 0$

controllability around y_γ : Crépeau 2001

$$y = y_1 + y_2 + \cdots$$

$$u = u_1 + u_2 + \cdots$$

$$y_{1t} + y_{1x} + y_{1xxx} = 0$$

$$y_1(t,0) = y_1(t,2\pi) = 0, \; y_{1x}(t,2\pi) = u_1(t)$$

$$y_{2t} + y_{2x} + y_{2xxx} = -y_1 y_{1x}$$

$$y_2(t,0) = y_2(t,2\pi) = 0, \; y_{2x}(t,2\pi) = u_2(t)$$

If one can find $u_{1\pm}$, $u_{2\pm}$ s.t.

if $y_{1\pm}(0,x) = y_{2\pm}(0,x) = 0$ then

$$y_{1\pm}(T,x) = 0$$

$$\int_0^{2\pi} y_{2\pm}(1-\cos x)\,dx = \pm 1$$

one gets local controllability

(intermediate value theorem; Brower fixed point theorem if more directions are missing)

After suitable transformation

The existence of $u_{1\pm}$, $u_{2\pm}$ can be searched by a computer with symbolic computation program (e.g. Maple)

$u_{1\pm}$ $u_{2\pm}$ do not exist

One needs to continue the expansion

$$y = y_1 + y_2 + y_3 + \cdots$$

$$u = u_1 + u_2 + u_3$$

$$y_{1t} + y_{1x} + y_{1xxx} = 0$$

$$y_1(t,0) = y_1(t,2\pi) = 0, \; y_{1x}(t,2\pi) = u_1(t)$$

$$y_{2t} + y_{2x} + y_{2xxx} = -y_1 y_{1x}$$

$$y_2(t,0) = y_2(t,2\pi) = 0 \quad y_{2x}(t,2\pi) = u_2(t)$$

$$y_{3t} + y_{3x} + y_{3xxx} = -y_1 y_{2x} - y_2 y_{1x}$$

$$y_3(t,0) = y_3(t,2\pi) = 0 \quad y_{3x}(t,2\pi) = u_3(t)$$

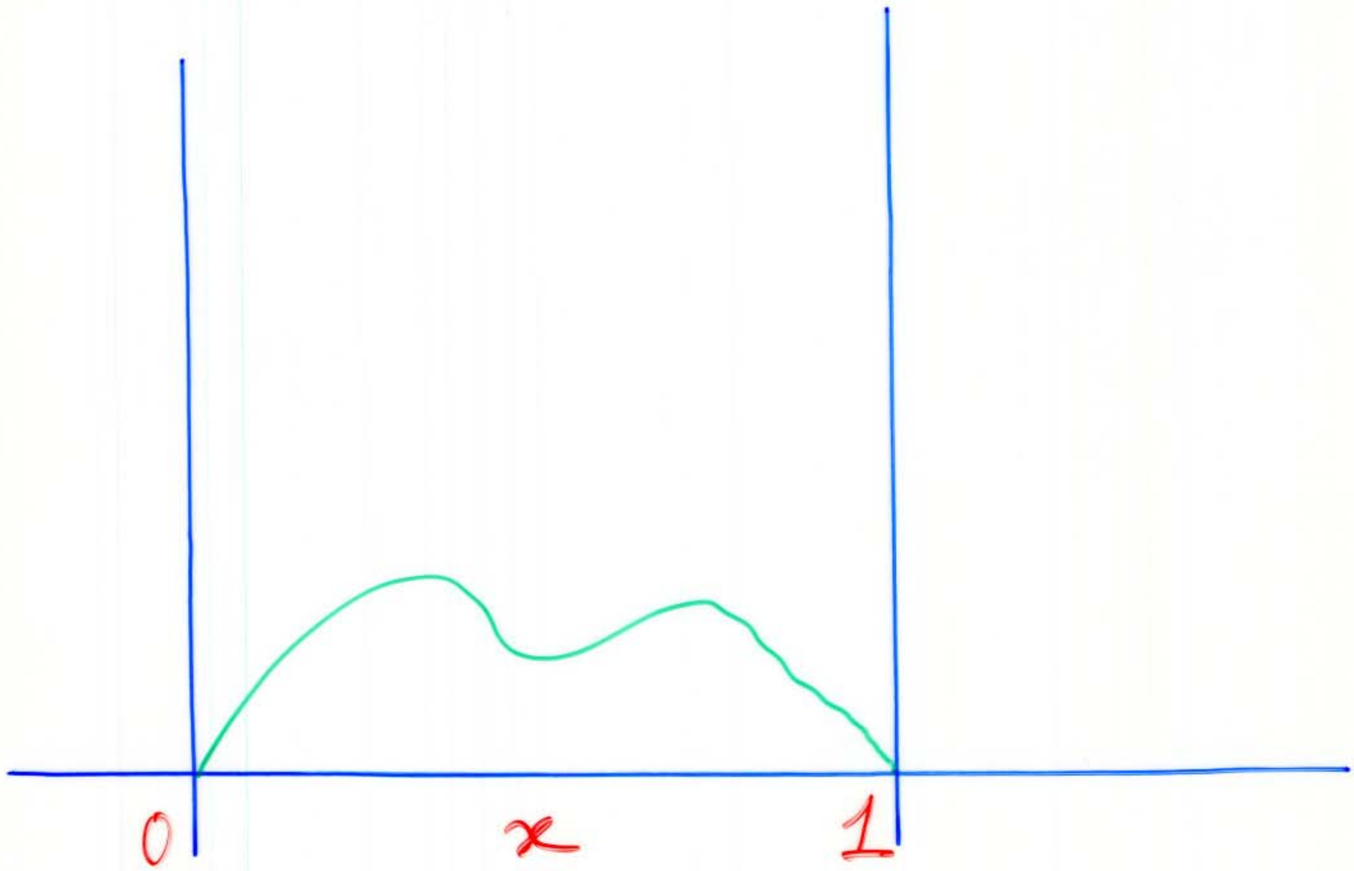
One wants

$$y_1(0,x) = y_1(T,x) = y_2(0,x) = y_2(T,x) = 0$$

$$y_3(0,x) = 0$$

$$\int_0^{2\pi} y_3(T,x)(1-\cos x)\,dx = \pm 1$$

THIS IS POSSIBLE
(FORMAL COMPUTATION)



$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} - u(t)\,x\,\psi$$

$$\psi(t,0) = \psi(t,1) = 0$$

$$\dot{S} = u(t)$$

$$\dot{D} = S(t)$$

$$\varphi_n(x) := \frac{1}{\sqrt{2}} \sin(\pi n x)$$

Thm (Beauchard – C 2008)

$\forall\, n \in \mathbb{N}^* \ \forall\, m \in \mathbb{N}^*$

$\forall\, D_0 \in \mathbb{R} \qquad \forall\, D_1 \in \mathbb{R}$

$\exists\ u \in H^1(0,T;\mathbb{R})$

such that

$$(\varphi_n, 0, D_0) \xrightarrow{\ u\ } (\varphi_m, 0, D_1)$$

+ local controllability

Prior result

P. Rouchon 2003

$$(\varphi_n, 0, D_0) \xrightarrow{u} (\varphi_n, 0, D_1)$$

is possible for the linearized control system around

$$\Psi_n(t,x) := e^{\frac{i\pi^2 n^2}{2} t} \varphi_n(x)$$

$S = 0$

$D = 0$

$u = 0$

Sketch of proof

n odd, m even

$$\psi^{\theta}(t,x) = \sqrt{1-\theta}\,\psi_n(t,x) + \sqrt{\theta}\,\psi_m(t,x)$$

Step 1

local controllability around

ψ^0 and ψ^1 for ψ ~~(S,D)~~

K. Beauchard 2005

<u>Beautiful paper</u>

Tools
- return method
- quasi-static deformations
- Nash-Moser

Step 2

In order to take care of (S, D)

$\longrightarrow$ power series expansion (as for KdV)

Step 3 local controllability around $(\varphi^{\theta}, 0, 0)$

Nash - Moser

+ power series expansion

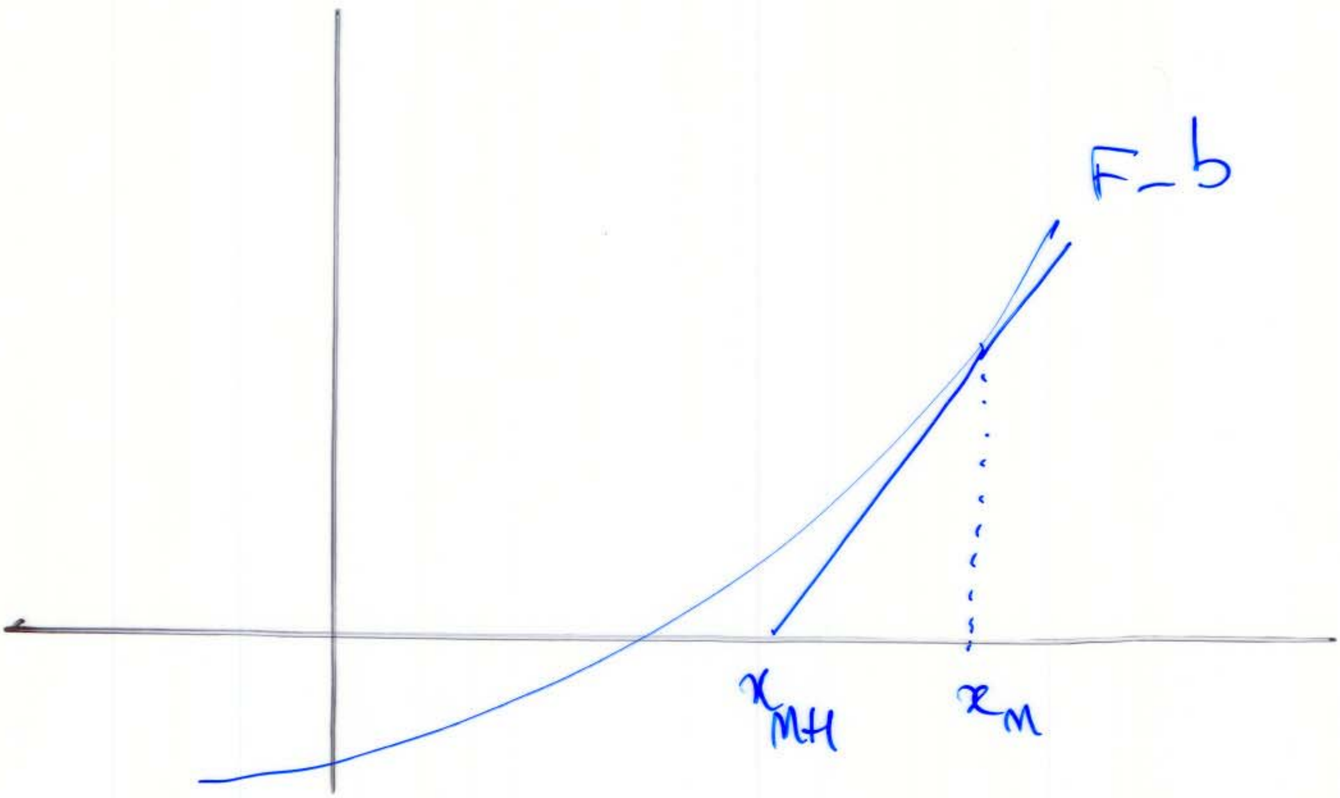
Nash-Moser

$$F(x_n) = b$$

$$x_{n+1} = x_n - S_n F'(x_n)^{-1}(F(x_n) - b)$$

Newton

S_n Nash-Moser



No <u>small-time</u> local controllability

$\exists\, \delta > 0$ $\exists\, T_0 > 0$ such that

$\forall\, D_1 \neq 0$

$$(\psi^1(0,\cdot), 0, 0) \longrightarrow (\psi^1(T_0,\cdot), 0, D_1)$$

with $|u(t)| \leq \delta$

is impossible

C. 2006