0.1 Galerkin time–stepping schemes

Discontinuous Galerkin methods

$$\int_{J_n} \left[(U', v) + (F(t,U), v)\right] dt + (U^{n-1+} - U^{n-1}, v^{n-1+}) = 0 \quad \forall v \in \mathcal{V}_q(J_n), \tag{1}$$

Besides the variational formulation (1), in the sequel we will also use a pointwise formulation of the dG(q) method.

The dG reconstruction (M. and Nochetto, Numer Math) $\hat{U} \in \mathcal{V}_{q+1}$ and $\hat{U}$ is continuous defined by

$$\int_{J_n} (\hat{U}', v)\, dt = \int_{J_n} (U', v)\, dt + (U^{n-1+} - U^{n-1}, v^{n-1+}) \quad \forall v \in \mathcal{V}_q(J_n), \tag{2}$$

Implicitly we require $\mathcal{U}^{n-1+} = U^{n-1}$

pointwise equation

$$\int_{J_n} \left[(\hat{U}', v) + (F(t, U), v)\right] dt = 0 \quad \forall v \in \mathcal{V}_q(J_n),$$

therefore

$$\hat{U}' + F(t, \hat{U}) = F(t, \hat{U}) - P_q F(t, U)$$

- Then : $\hat{U}$ and U coincide at the $q+1$ Radau points in the interval J_n, i.e., at the points $t^{n,i} := t^{n-1} + k_n \tau_i$, $i = 1, \dots, q+1$, with $0 < \tau_1 < \cdots < \tau_{q+1} = 1$ the nodes of the quadrature formula that integrates exactly in $[0, 1]$ polynomials of degree at most $2q$.

- Note for later use that if I_q is the interpolation operator at $t^{n,i}$, $i = 1, \dots, q+1$, i.e., $(I_q \varphi)(t^{n,i}) = \varphi(t^{n,i})$, $i = 1, \dots, q+1$, we have

$$U = I_q \hat{U}$$

and

dG = cG + Interpolation

Then the (continuous) reconstruction satisfies the equation

$$\int_{J_n} \left[(\hat{U}', v) + (F(t, I_q \hat{U}), v)\right] dt = 0 \quad \forall v \in \mathcal{V}_q(J_n), \tag{3}$$

$n = 1, \ldots, N.$

or

$$\hat{U}'(t) + P_q F(t, (I_q \hat{U})(t)) = 0 \quad \forall t \in J_n,$$

Continuous Galerkin + C-Runge Kutta

We cast these methods in a wider class of schemes formulated in a unified form with the aid of a projection operator Π_ℓ onto $\mathcal{H}_\ell(J_n), n = 1, \ldots, N$. The time discrete approximation U is defined as follows: We seek $U \in \mathcal{V}_q$, U continuous satisfying the initial condition $U(0) = u^0$, such that

$$U'(t) + \Pi_{q-1} AU(t) = \Pi_{q-1} f(t) \quad \forall t \in J_n,$$

for $n = 1, \ldots, N$. Since all terms in this equation belong to $\mathcal{V}_{q-1}(J_n)$, This is equivalent to the Galerkin formulation

$$\int_{J_n} \big[(U', v) + (\Pi_{q-1} AU, v)\big]\, dt = \int_{J_n} (\Pi_{q-1} f, v)\, dt \quad \forall v \in \mathcal{V}_{q-1}(J_n),$$

- continuous Galerkin method corresponds to the choice $\Pi_{q-1} := P_{q-1}$, with P_ℓ denoting the L^2 orthogonal projection operator onto $\mathcal{H}_\ell(J_n)$;

- Runge–Kutta collocation methods constitute the most important class of time–stepping schemes described by this formulation. We see later that all RK–C methods can be obtained by choosing $\Pi_{q-1} := I_{q-1}$, with I_{q-1} denoting the interpolation operator by elements of $\mathcal{V}_{q-1}(J_n)$ at the nodes $t^{n-1} + \tau_i k_n, i = 1, \dots, q, n = 1, \dots, N$, with appropriate $0 \le \tau_1 < \cdots < \tau_q \le 1$.

- It is well known that RK Gauss–Legendre schemes are directly connected to continuous Galerkin methods. A first conclusion, probably not observed before, is that all RK–C methods can be obtained by applying appropriate numerical quadrature to continuous Galerkin methods.

Reconstruction

We define reconstructions $\hat{U} \in \mathcal{H}_{q+1}$ of the approximate solution U via appropriate projection operators $\hat{\Pi}_q$ onto $\mathcal{H}_q(J_n), n = 1, \dots, N$. In the error estimates we will implicitly assume additional spatial smoothness for $\hat{U}$ when this is needed. A fundamental property we require for $\hat{\Pi}_q$ is that

$$\int_{J_n} \hat{\Pi}_q \varphi(s)\, ds = \int_{J_n} \Pi_{q-1} \varphi(s)\, ds \quad \forall \varphi \in C(J_n; H), \tag{4}$$

for $n = 1, \dots, N$. We define the reconstruction $\hat{U} \in \mathcal{H}_{q+1}(J_n)$ of U by

$$\hat{U}(t) := U(t^{n-1}) - \int_{t^{n-1}}^{t} [AU(s) - \hat{\Pi}_q f(s)]\, ds \quad \forall t \in J_n\,. \tag{5}$$

Obviously, $\hat{U}(t^{n-1}) = U(t^{n-1})$. Furthermore,

$$\begin{aligned} \hat{U}(t^n) &= U(t^{n-1}) - \int_{t^{n-1}}^{t^n} [A\hat{\Pi}_q U(s) - \hat{\Pi}_q f(s)]\, ds \\ &= U(t^{n-1}) - \int_{t^{n-1}}^{t^n} [A\Pi_{q-1} U(s) - \Pi_{q-1} f(s)]\, ds; \end{aligned}$$

We conclude

$$\hat{U}(t^n) = U(t^{n-1}) + \int_{t^{n-1}}^{t^n} U'(s)\,ds,$$

i.e., $\hat{U}(t^n) = U(t^n)$, and conclude that $\hat{U}$ is continuous in $[0, T]$.

It easily follows that $\hat{U}$ satisfies the following pointwise equation

$$\hat{U}'(t) + AU(t) = \hat{\Pi}_q f(t) \quad \forall t \in J_n; \tag{6}$$

Going back to the general class of schemes

All the above methods can be written in the form

$$\int_{J_n} \left[(\hat{U}', v) + (\Pi_{q-1} F(t, \Pi^\star \hat{U}), v)\right] dt = 0 \quad \forall v \in \mathcal{V}_q(J_n), \tag{7}$$

where Π_{q-1} and $\Pi^\star$ are appropriate projection or interpolation operators.

Perturbed Collocation Methods, Norset and Wanner

Superconvergence for C-RK

After the seminal thesis of M. Crouzeix the classical order of convergence of RK schemes can be considered as superconvergence order for PDEs.

We call it here superorder.

Theorem [Superorder for linear equations] Let the superorder of a q-stage RK-C method satisfy $p \in \{q+2, \dots, 2q\}$. Then the following a posteriori error estimate is valid at the nodes $\{t^n\}_{n=1}^N$ for linear equations

$$|\hat{e}(t^n)| \leq C_I L_n \big(\mathcal{E}_1 + \mathcal{E}_2\big),$$

with

$$\mathcal{E}_1 = \max_{1 \leq m \leq n-1} \Big(k_m^{p-q-1} |A^{p-q-2}\big(R_{\hat{U}} + R_f\big)|_{L^\infty(J_m)}, \\ k_n^{p-q-2} \int_{J_n} |A^{p-q-2}\big(R_{\hat{U}} + R_f\big)| \, dt \Big),$$

$$\mathcal{E}_2 = \sum_{j=0}^{p-q-2} \max_{1 \leq m \leq n-1} \Big(k_m^j |A^{j-1}\big(f - \hat{I}_{p-j-1} f\big)|_{L^\infty(J_m)}, \\ k_n^{j-1} \int_{J_n} |A^{j-1}\big(f - \hat{I}_{p-j-1} f\big)| \, dt \Big).$$

The order of these estimators is p, provided the solution is smooth.

Related works

- high-order Discontinuous Galerkin (Energy) —Nonlinear Problems, Angle bounded operators , Conditional estimates for the Minimal Surface equation—M. & R. Nochetto (Numer Math)

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Questions....

- Fully Discrete Nonlinear Problems : Conditional estimates
- high-order Runge-Kutta methods combined with space discretization....
- estimates for popular open problems with singular behaviour, exponential constants....
- notion of convergence of adaptive algorithms, convergence.