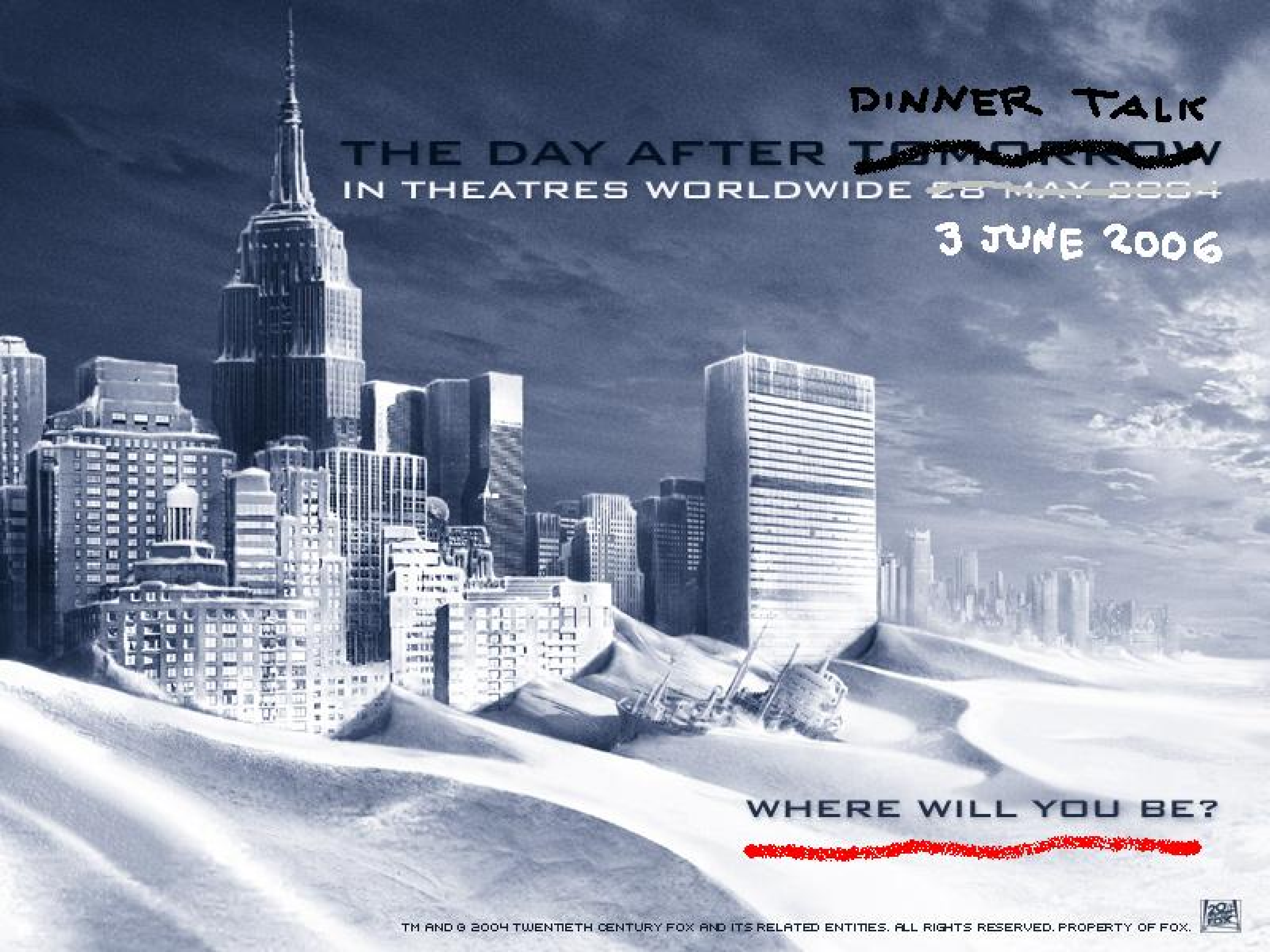
Coupling of Boundary Element and Discontinuous Galerkin Methods

Francisco–Javier Sayas

Departamento de Matemática Aplicada
Centro Politécnico Superior
Universidad de Zaragoza (Spain)

$1\frac{1}{2}$ Journées Crouzeix
Guidel, 2nd-3rd June 2006


DINNER TALK
THE DAY AFTER ~~TOMORROW~~
IN THEATRES WORLDWIDE ~~28 MAY 2004~~
3 JUNE 2006
WHERE WILL YOU BE?
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My coauthors are...

- Rommel A. Bustinza
- Gabriel N. Gatica

both at the ...

Departamento de Ingeniería Matemática
Universidad de Concepción, Chile

Our aims

- Showing how/if Discontinuous Galerkin Methods can manage exact absorbing boundary conditions (non–local)
- Showing how well/bad (L)DG can be used in some thermal scattering problems.
- Like storks,... flying south during the wintertime.

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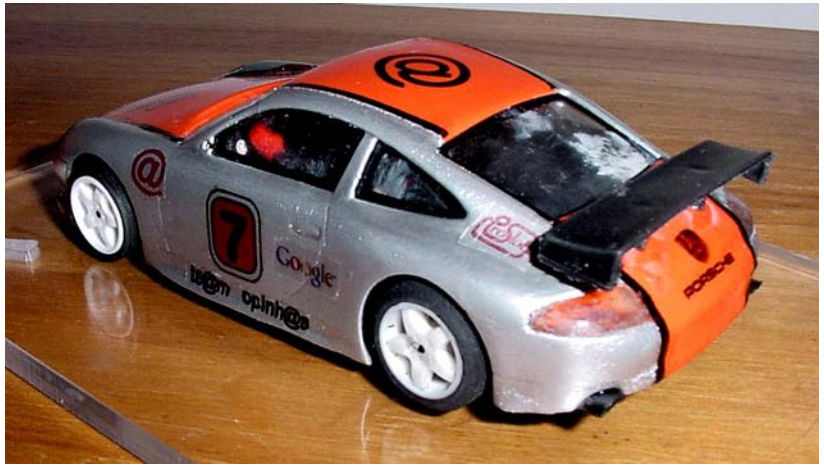
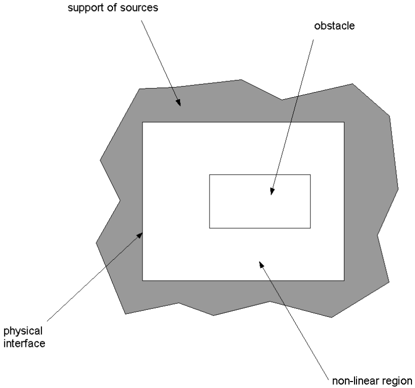
THE MODEL PROBLEM



Geometrical setting and governing equations



Boundary of the obstacle:

$u = g_0$

Non–linear region:

$\mathrm{div}\,\mathbf{a}(\,\cdot\,,\nabla u) + f = 0$

Linear region:

$\Delta u + f = 0$

Interface:

$u^- = u^+ + g_1$

$\mathbf{a}(\,\cdot\,,\nabla u^-)\cdot\mathbf{n} = \partial_{\mathbf{n}} u^+ + g_2$

f with compact support

$u = \mathcal{O}(1)$ at ∞

... or $\mathcal{O}(1/r)$ in 3D.

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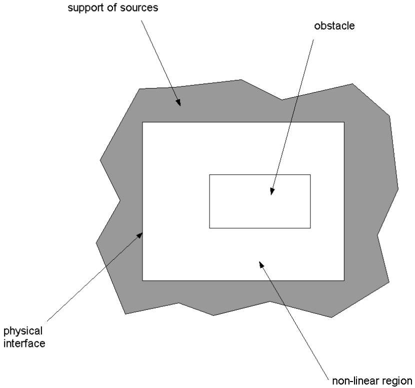

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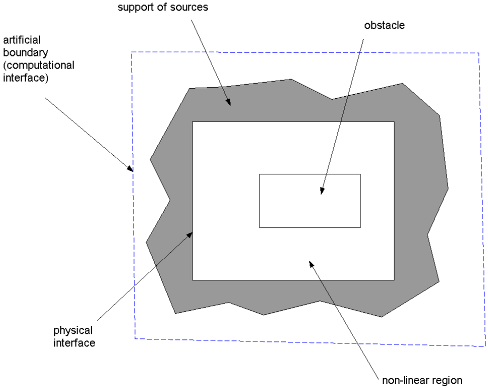
Artificial boundary



...

Bounded linear region:

$\Delta u + f = 0$

New interface:

$u^{\text{int}} = u^{\text{ext}}$

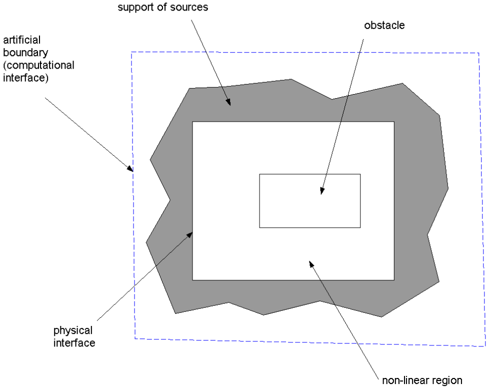
$\partial_{\mathbf{n}} u^{\text{int}} = \partial_{\mathbf{n}} u^{\text{ext}}$

Unbounded linear region:

$\Delta u = 0$

...

Artificial boundary



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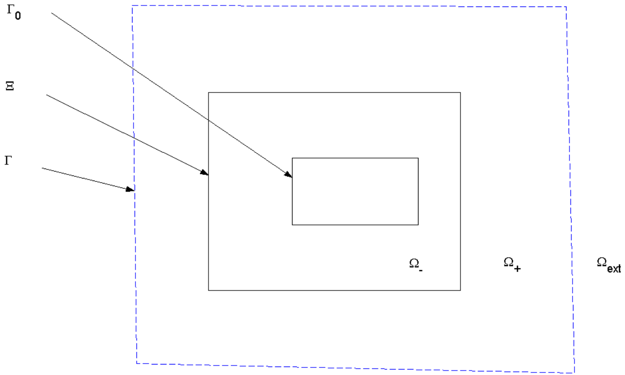
Notations



$$\Omega := \Omega_- \cup \Xi \cup \Omega_+$$

Hypotheses

- $g_0 \in H^{1/2}(\Gamma_0)$
- $g_1 \in H^{1/2}(\Xi), \qquad g_2 \in L^2(\Xi)$
- Carathéodory conditions for $\mathbf{a}(\mathbf{x},\xi)$ and $D_\xi \mathbf{a}(\mathbf{x},\xi)$
- Growth conditions for $\mathbf{a}$ and $D_\xi \mathbf{a}$:

$$|\mathbf{a}(\mathbf{x},\xi)| \le C|\xi| + D(\mathbf{x}), \quad D \in L^2(\Omega_-).$$

$$|D_\xi \mathbf{a}(\mathbf{x},\xi)| \le C.$$

- Uniform ellipticity for $D_\xi \mathbf{a}$
- $f \in L^2(\Omega)$

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Interior (three–field formulation)

Equations

$$\begin{aligned} u &= g_0, && \text{on } \Gamma_0 \\ \boldsymbol{\sigma} &= \mathbf{a}(\,\cdot\,,\boldsymbol{\theta}), && \text{in } \Omega_- \\ \boldsymbol{\theta} &= \nabla u, && \text{in } \Omega_- \\ \operatorname{div}\boldsymbol{\sigma} + f &= 0, && \text{in } \Omega_- \end{aligned}$$

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with interface conditions on Ξ

$$u^- = u^+ + g_1, \qquad \boldsymbol{\sigma}^- \cdot \mathbf{n} = \boldsymbol{\sigma}^+ \cdot \mathbf{n} + g_2$$

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BOUNDARY MATTERS

Fundamental solution

$$\Phi(\mathbf{x},\mathbf{y}) = \begin{cases} -1/(2\pi)\,\log|\mathbf{x}-\mathbf{y}| & \text{2D} \\ 1/(4\pi\,|\mathbf{x}-\mathbf{y}|), & \text{3D} \end{cases}$$

$$-\Delta u = 0, \quad \text{in } \Omega_{\text{ext}}, \qquad u(\infty) = \begin{cases} \mathcal{O}(1) & \text{2D} \\ \mathcal{O}(1/r) & \text{3D} \end{cases}$$

Third Green's Theorem

$$\Theta u(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{ds}(\mathbf{y}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}u(\mathbf{y})\mathrm{ds}(\mathbf{y})(+c)$$

$$\Theta = \begin{cases} 1 & \mathbf{x} \quad \text{outside} \\ 1/2 & \mathbf{x} \quad \text{on the boundary} \\ 0 & \mathbf{x} \quad \text{inside} \end{cases}$$

The constant appears in the two dimensional case.

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The constant appears in the two dimensional case.

Exterior points ($\Theta = 1$)

$$u(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}s(\mathbf{y}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}u(\mathbf{y})\mathrm{d}s(\mathbf{y})$$

representation formula

But at the boundary ($\Theta = 1/2$)

$$\begin{array}{rcccc} \frac{1}{2}u(\mathbf{x}) & = & \int_\Gamma \partial_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}s(\mathbf{y}) & - & \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}u(\mathbf{y})\mathrm{d}s(\mathbf{y}) \\ & =: & \mathcal{K}u & - & \mathcal{V}\partial_{\mathbf{n}}u \end{array}$$

the Cauchy data are related (= integral equation)

(Forget the additional constant and other conditions; as if we were solving $-\Delta u + u = 0$)

An integral identity on Γ

$$\mathcal{V}\partial_{\mathbf{n}}u + (\tfrac{1}{2} - \mathcal{K})u = 0$$

Solve:

$$\mathcal{V}\gamma + (\tfrac{1}{2} - \mathcal{K})\xi = 0$$

... then $\gamma = \partial_{\mathbf{n}} u$

- V is elliptic in $H^{-1/2}(\Gamma)$ (good for Galerkin!) ... (forget the problematic constants of the Laplacian, please)
- ξ appears under the action of an integral operator
- for coupling problems, ξ comes all discretized

Dirichlet–to–Neumann Operator $(u = \xi \text{ on } \Gamma)$

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What do engineering books say?

See: Beer (01), Gaul, Kögel, Wagner (03). See perhaps: Brebbia & Dominguez (92)

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	X_h(disc)	Y_h(cont)
X_h	**V**	$\frac{1}{2}\mathbf{I} - \mathbf{K}$
Y_h	**0**	**I**

- **V** : Galerkin for elliptic operator
- $\mathbf{I}\varphi = \xi$ rediscretizes data
- The $L^2(\Gamma)$–orth. projection onto $H^{1/2}(\Gamma)$ has to be stable, which is the case when it's $H^1(\Gamma)$ stable, as in Crouzeix & Thomée (87) and related work

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Another option

	X_h(disc)	Y_h(cont)
X_h	$\mathbf{V}$	$\frac{1}{2}\mathbf{I} - \mathbf{K}$
Z_h(disc)	$\mathbf{0}$	$\mathbf{I}$

There's now an inf–sup (discrete BB) condition to be satisfied.

Dual meshes. See: Steinbach (02), Rapún & FJS (06). See also: fluid mechanics FE literature, finite volume cells

$$\begin{aligned} \gamma & & &= \lambda & &\text{in } H^{-1/2}(\Gamma) \\ \mathcal{V}\gamma &+ (\tfrac{1}{2} - \mathcal{K})\varphi & &= 0, & &\text{in } H^{1/2}(\Gamma) \end{aligned}$$

	Z_h(disc)	(Y_h)(cont)
Y_h	$\mathbf{I}^\top$	$\mathbf{0}$
(Y_h)	$\mathbf{V}$	$\frac{1}{2}\mathbf{I} - \mathbf{K}$

New difficulties:

- $\frac{1}{2} - \mathcal{K}$ is not identity + compact
- it's identity + small + compact (known since long ago in $L^2(\Gamma)$; see Steinbach & Wendland (01) in $H^{1/2}(\Gamma)$)
- not very helpful when discretizing
- and elasticity is out of the question

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What shall we do?

We go back to Green's 3rd Theorem

$$u(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{ds}(\mathbf{y}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}u(\mathbf{y})\mathrm{ds}(\mathbf{y})$$

and take the normal derivative

$$\begin{aligned}
\partial_{\mathbf{n}}u(\mathbf{x}) &= \partial_{\mathbf{n}(\mathbf{x})}\int_\Gamma \partial_{\mathbf{n}(\mathbf{y})}\Phi(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{ds}(\mathbf{y}) \\
&\qquad + \tfrac{1}{2}\partial_{\mathbf{n}}u(\mathbf{x}) - \int_\Gamma \partial_{\mathbf{n}(\mathbf{x})}\Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}u(\mathbf{y})\mathrm{ds}(\mathbf{y}) \\
&=: -\mathcal{W}u + \tfrac{1}{2}\partial_{\mathbf{n}}u - \mathcal{K}'\partial_{\mathbf{n}}u
\end{aligned}$$

A new identity/beginning

$$\mathcal{W}u + (\tfrac{1}{2} + \mathcal{K}')\partial_{\mathbf{n}} u = 0$$

- $\mathcal{W}$ is elliptic (hence the sign!) (the constants,please!)
- We can proceed as before ... still with the problem of stabilising a discrete identity operator (now in $H^{-1/2}(\Gamma)$)...
- ... and since we dared to deal with $\mathcal{W}$ (hypersingular), why not using the whole package? $(\mathcal{V}, \mathcal{W}, \mathcal{K}, \mathcal{K}')$

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Two identities...

$$\begin{array}{rcccl} \mathcal{W}\varphi & + & (\frac{1}{2}+\mathcal{K}')\gamma & = & 0 \\ (\frac{1}{2}-\mathcal{K})\varphi & + & \mathcal{V}\gamma & = & 0 \end{array}$$

... become two equations

$$\begin{array}{rcccl} \mathcal{W}\varphi & + & (-\frac{1}{2}+\mathcal{K}')\gamma & = & -\lambda \\ (\frac{1}{2}-\mathcal{K})\varphi & + & \mathcal{V}\gamma & = & 0 \end{array}$$

Elliptic system, very apt for Galerkin.

See: Costabel (87), Han (90). See also (for 1–equation coupling): Johnson & Nédélec (80), Brezzi & Johnson (79). See even: Zienkiewicz, Kelly and Bettess (77)

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... become two equations

$$\begin{array}{rcrcl} \mathcal{W}\varphi & + & (-\frac{1}{2}+\mathcal{K}')\gamma & = & -\lambda \\ (\frac{1}{2}-\mathcal{K})\varphi & + & \mathcal{V}\gamma & = & 0 \end{array}$$

Elliptic system, very apt for Galerkin.

See: Costabel (87), Han (90). See also (for 1–equation coupling): Johnson & Nédélec (80), Brezzi & Johnson (79). See even: Zienkiewicz, Kelly and Bettess (77)

The operator *NtD*

$$\begin{array}{rcccl} \langle \mathcal{W}\varphi, \psi \rangle & + & \langle (-\tfrac{1}{2} + \mathcal{K}')\gamma, \psi \rangle & = & -\langle \lambda, \psi \rangle, \quad \forall \psi \\ \langle \mu, (\tfrac{1}{2} - \mathcal{K})\varphi \rangle & + & \langle \mu, \mathcal{V}\gamma \rangle & = & 0, \quad \forall \mu \end{array}$$

$$\lambda \mapsto (\gamma, \varphi) \mapsto \varphi := NtD(\lambda)$$

$$\|\lambda\|_{-1/2,\Gamma} = \sup_{\psi} \frac{\langle \lambda, \psi \rangle}{\|\psi\|_{1/2,\Gamma}} \leq C \left[\|\varphi\|_{1/2,\Gamma} + \|\gamma\|_{-1/2,\Gamma} \right]$$

$$\begin{aligned} -\langle \lambda, NtD(\lambda) \rangle &= -\langle \lambda, \varphi \rangle = \langle \mathcal{W}\varphi, \varphi \rangle + \langle (-\tfrac{1}{2} + \mathcal{K}')\gamma, \varphi \rangle \\ &= \langle \mathcal{W}\varphi, \varphi \rangle + \langle \mathcal{V}\gamma, \gamma \rangle \geq C \|\lambda\|^2_{-1/2,\Gamma} \end{aligned}$$

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The discrete operator NtD_h

$Y_h \subset H^{1/2}(\Gamma), Z_h \subset H^{-1/2}(\Gamma)$. Hence Y_h are continuous elements and Z_h discontinuous ones.

$\varphi_h \in Y_h, \gamma_h \in Z_h$

$$\begin{array}{rcl} \langle \mathcal{W}\varphi_h, \psi_h\rangle + \langle(-\tfrac{1}{2}+\mathcal{K}')\gamma_h, \psi_h\rangle &=& -\langle \lambda, \psi_h\rangle, \quad \forall \psi_h \in Y_h \\ \langle \mu_h, (\tfrac{1}{2}-\mathcal{K})\varphi_h\rangle + \langle \mu_h, \mathcal{V}\gamma_h\rangle &=& 0, \quad \forall \mu_h \in Z_h \end{array}$$

$$\lambda \mapsto (\gamma_h, \varphi_h) \mapsto \varphi_h := NtD_h(\lambda)$$

$$|\lambda|_h := \sup_{\psi_h \in Y_h} \frac{\langle \lambda, \psi_h\rangle}{\|\psi_h\|_{1/2,\Gamma}} \le \|\lambda\|_{-1/2,\Gamma}$$

$$\|\varphi_h\|_{1/2,\Gamma} + \|\gamma_h\|_{-1/2,\Gamma} \lesssim |\lambda|_h, \qquad -\langle \lambda, NtD_h(\lambda)\rangle \gtrsim |\lambda|_h^2$$

Taking care of constants

In two dimensions...

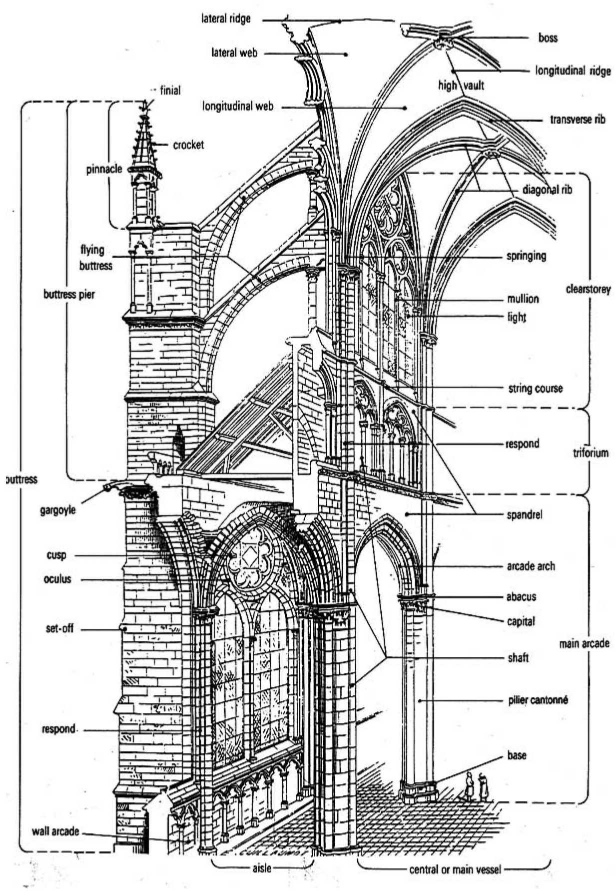
$$\lambda = \gamma \in H_0^{-1/2}(\Gamma), \qquad \text{meaning} \quad \int_\Gamma \lambda = 0$$

and

$$u_\Gamma = \varphi + \kappa, \qquad \varphi \in H_0^{1/2}(\Gamma), \qquad \text{i.e.} \quad \int_\Gamma \varphi = 0.$$

To know behaviour at infinity we have to know $\int_\Gamma u$.
All the preceding results (continuous/discrete) are easily adapted.

THE INTERIOR



Why DG?

- Ask a real expert
- Complicated geometries where non–regular meshes fit better.
- Different degrees, simpler refining strategies (hanging nodes)
- Promising parallelization capabilities
- Possibility of handling non–linearities at an element level.

For LDG, see: Cockburn & Shu (89) and related work. See also: Bustinza & Gatica (04, 05). See especially: Arnold, Brezzi, Cockburn, Marini (01/02)

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Geometric aspects of DG

- Separated triangulations of Ω_- and Ω_+.
- Each one with: shape regular triangles, possible hanging nodes, (asymptotically) bounded number of neighbours, (asymp) no slipping interfaces, etc.
- $\mathcal{T}_h \ni K \mapsto \mathbb{P}(K)$: polynomial space for scalar fields with (asymptotically) bounded degree (no $h - p$ here and now)
- $\mathbf{P}(K)$: vector polynomials (of same degree as $\mathbb{P}(K)$ or one less), ensuring that $\nabla\mathbb{P}(K) \subset \mathbf{P}(K)$.
- $V_h := \prod_K \mathbb{P}(K)$ space for scalar unknowns
- $\mathbf{\Sigma}_h := \prod_K \mathbf{P}(K)$ for vector unknowns: $\nabla_h V_h \subset \mathbf{\Sigma}_h$

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Local form of LDG methods

$$u_h \in V_h := \prod_K \mathbb{P}(K), \qquad \boldsymbol{\sigma}_n, \boldsymbol{\theta}_h \in \boldsymbol{\Sigma}_h := \prod_K \mathbf{P}(K).$$

False trace & normal flux on the set of sides: $\widehat{u}$, $\widehat{\boldsymbol{\sigma}}$.

$$\begin{aligned}
\boldsymbol{a}(\,\cdot\,,\boldsymbol{\theta}) = \boldsymbol{\sigma} \qquad & \int_K \boldsymbol{a}(\,\cdot\,,\boldsymbol{\theta}_h)\cdot\boldsymbol{\zeta} = \int_K \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta} \\
\nabla u = \boldsymbol{\theta} \qquad & \int_K \boldsymbol{\theta}_h \cdot \boldsymbol{\tau} + u_h\,(\mathrm{div}_h \boldsymbol{\tau}) = \int_{\partial K} \widehat{u}\, \boldsymbol{\tau}\cdot\mathbf{n}, \\
-\mathrm{div}\,\boldsymbol{\sigma} = f \qquad & \int_K \boldsymbol{\sigma}_h \cdot \nabla v = \int_K f\, v + \int_{\partial K} \widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}\, v
\end{aligned}$$

$$\forall \boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbf{P}(K),\ v \in \mathbb{P}(K)$$

$\mathcal{E}_h^{\rm int}$:=set of sides not on boundaries or interfaces. When needed, any trace can be understood elementwise.

Averaging operator

$$\{\,\cdot\,\} : H^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h^{\rm int}), \qquad \{\,\cdot\,\} : \mathbf{H}^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathcal{E}_h^{\rm int})$$

Jumps

$$[\,\cdot\,] : H^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathcal{E}_h^{\rm int}) \qquad [u] = u_1\mathbf{n}_1 + u_2\mathbf{n}_2$$

$$[\,\cdot\,] : \mathbf{H}^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h^{\rm int}) \qquad [\sigma] = \sigma_1 \cdot \mathbf{n}_1 + \sigma_2 \cdot \mathbf{n}_2$$

Jumps and averages

$\mathcal{E}_h^{\text{int}}$:=set of sides not on boundaries or interfaces. When needed, any trace can be understood elementwise.

Averaging operator

$$\{\cdot\} : H^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h^{\text{int}}), \qquad \{\cdot\} : \mathbf{H}^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathcal{E}_h^{\text{int}})$$

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Why the dance of scalars and vectors?

Discrete divergence theorem

$$\int_{\mathcal{O}} \nabla_h v \cdot \boldsymbol{\tau} + \int_{\mathcal{O}} v \,\mathrm{div}_h \boldsymbol{\tau} = \int_{I_h} ([v] \cdot \{\boldsymbol{\tau}\} + \{v\}[\boldsymbol{\tau}]) + \int_{\partial\mathcal{O}} (v\,\mathbf{n})\,\boldsymbol{\tau}$$

In particular: if $\boldsymbol{\tau}$ is smooth and compactly supported, this gives the distributional gradient of a piecewise smooth function.

Numerical fluxes I: a false trace

$$\boldsymbol{\beta} \in \prod_e \mathbb{P}_0(e), \qquad \boldsymbol{\beta} \parallel \mathbf{n}, \qquad |\boldsymbol{\beta}| \lesssim 1.$$

$$\begin{array}{rll} \widehat{u}: & H^1(\mathcal{T}_h) & \longrightarrow \quad L^2(\mathcal{E}_h) \\ & \times & \\ & L^2(\Gamma_0) \ni g_0 & \\ & \times & \\ & L^2(\Xi) \ni g_1 & \end{array}$$

- e interior side: average with convected jump

$$\widehat{u} = \{u\} + \beta \cdot [u]$$

- $e \subset \Gamma_0$: Dirichlet datum

$$\widehat{u} = g_0$$

- $e \subset \Xi_-$ (interface seen from inside): Dirichlet datum ($u_- = u_+ + g_1$)

$$\widehat{u} = u_+ + g_1$$

- $e \subset \Xi_+$ (interface seen from outside): Neumann side

$$\widehat{u} = u$$

- $e \subset \Gamma$ (exterior boundary): Neumann condition

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Numerical fluxes II: a false normal flux

$$\alpha \in \prod_e \mathbb{P}_0(e), \qquad h\,\alpha \approx 1.$$

$$\begin{array}{rll} \widehat{\boldsymbol{\sigma}} : & H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) & \longrightarrow \quad \mathbf{L}^2(\mathcal{E}_h) \\ & \times & \\ & L^2(\Gamma_0) \ni g_0 & \\ & \times & \\ & L^2(\Xi) \ni g_1 & \\ & \times & \\ & L^2(\Xi) \ni g_2 & \\ & \times & \\ & L^2(\Gamma) \ni \lambda & \end{array}$$

- e interior side: average of fluxes minus convected jump minus penalization

$$\widehat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}\} - [\boldsymbol{\sigma}]\,\boldsymbol{\beta} - \alpha\,[u]$$

- $e \subset \Gamma_0$: penalized flux

$$\widehat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \alpha\,(u - g_0)\,\mathbf{n}$$

- $e \subset \Xi_-$ (Dirichlet view of the interface):

$$\widehat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_- - \alpha\,([u] - g_1\mathbf{n})$$

- $e \subset \Xi_+$ (Neumann view): ($\boldsymbol{\sigma}_- \cdot \mathbf{n} = \boldsymbol{\sigma}_+ \cdot \mathbf{n} + g_2$)

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Once again... the equations

$$\int_K \mathbf{a}(\,\cdot\,,\boldsymbol{\theta}_h)\cdot\boldsymbol{\zeta} - \int_K \boldsymbol{\sigma}_h\cdot\boldsymbol{\zeta} = 0$$

$$\int_K \boldsymbol{\theta}_h\cdot\boldsymbol{\tau} + \left\{ \begin{array}{l} \int_K u_h\,(\mathrm{div}_h\boldsymbol{\tau}) \\ -\int_{\partial K} \widehat{u}\,\boldsymbol{\tau}\cdot\mathbf{n} \end{array} \right\} = 0$$

$$\left\{ \begin{array}{l} \int_K \boldsymbol{\sigma}_h\cdot\nabla v \\ \int_{\partial K} \widehat{\boldsymbol{\sigma}}_{flow}\cdot\mathbf{n}\,v \end{array} \right\} + \int_{\partial K} \widehat{\boldsymbol{\sigma}}_{pen}\cdot\mathbf{n}\,v = \int_K f\,v$$

- Misleadingly mixed–looking problem! What counts here is ellipticity.
- Fluxes are the interelement connections and include information on BC.

Once again... the equations

$$
\begin{aligned}
&\int_K \mathbf{a}(\cdot\,,\theta_h)\cdot\zeta - \int_K \boldsymbol{\sigma}_h \cdot \zeta && = 0\\
&\int_K \theta_h \cdot \boldsymbol{\tau} \qquad + \left\{ \begin{array}{l} \int_K u_h\,(\mathrm{div}_h \boldsymbol{\tau}) \\ -\int_{\partial K} \widehat{u}\,\boldsymbol{\tau}\cdot\mathbf{n} \end{array} \right\} && = 0\\
&\left\{ \begin{array}{l} \int_K \boldsymbol{\sigma}_h \cdot \nabla v \\ \int_{\partial K} \widehat{\boldsymbol{\sigma}}_{flow}\cdot\mathbf{n}\,v \end{array} \right\} + \int_{\partial K} \widehat{\boldsymbol{\sigma}}_{pen}\cdot\mathbf{n}\,v && = \int_K f\,v
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Correcting the piecewise gradient

$$\nabla_h^* u := \nabla_h u - S_h(u)$$

$$
\begin{aligned}
& S_h(u) \in \boldsymbol{\Sigma}_h \\
& \int_\Omega S_h(u) \cdot \boldsymbol{\tau}_h = \int_{I_h} [u] \cdot (\{\boldsymbol{\tau}_h\} - [\boldsymbol{\tau}_h]\beta) \\
& \qquad + \int_{\Gamma_0} u \, (\boldsymbol{\tau}_h \cdot \mathbf{n}) + \int_{\Xi} [u] \, (\boldsymbol{\tau}_h^- \cdot \mathbf{n}), \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h
\end{aligned}
$$

Solving and substituting

- The second group of equations states that

$$\boldsymbol{\theta}_h = \nabla_h^* u_h + \boldsymbol{g}_h$$

where $\boldsymbol{g}_h$ takes care of g_0 and g_1.

- The first one asserts that

$$\int_\Omega a(\,\cdot\,, \nabla_h^* u_h + \boldsymbol{g}_h) \cdot \zeta_h = \int_\Omega \boldsymbol{\sigma}_h \cdot \zeta_h$$

(true in particular for $\zeta_h = \nabla_h^* v_h$).

- Finally, the third block says

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h^* v_h + \alpha(u_h, v_h) = \int_\Gamma \lambda v_h + \int_\Omega f\, v_h + \text{B \& T terms}$$

where

$$\alpha(u,v) = \int_{I_h} \alpha\, [u] \cdot [v] + \int_{\Gamma_0} \alpha\, u\, v + \int_{\Xi} \alpha\, [u]\, [v]$$

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Solving and substituting

- The second group of equations states that

$$\boldsymbol{\theta}_h = \nabla_h^* u_h + \boldsymbol{g}_h$$

where $\boldsymbol{g}_h$ takes care of g_0 and g_1.

- The first one asserts that

$$\int_\Omega a(\,\cdot\,, \nabla_h^* u_h + \boldsymbol{g}_h) \cdot \zeta_h = \int_\Omega \boldsymbol{\sigma}_h \cdot \zeta_h$$

(true in particular for $\zeta_h = \nabla_h^* v_h$).

- Finally, the third block says

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h^* v_h + \alpha(u_h, v_h) = \int_\Gamma \lambda v_h + \int_\Omega f\, v_h + \text{B \& T terms}$$

where

$$\alpha(u, v) = \int_{I_h} \alpha\, [u] \cdot [v] + \int_{\Gamma_0} \alpha\, u\, v + \int_{\Xi} \alpha\, [u]\, [v]$$

The so–called primal formulation

The LDG equations are equivalent to...

$$\int_\Omega \mathbf{a}(\,\cdot\,,\nabla_h^* u_h + \boldsymbol{g}_h)\cdot\nabla_h^* v_h + \alpha(u_h,v_h) \quad = \quad \int_\Gamma \lambda v_h + \int_\Omega f\, v_h$$

$+$B & T terms

$$B_h(u,v) := \int_\Omega \mathbf{a}(\,\cdot\,,\nabla_h^* u + \boldsymbol{g}_h)\cdot\nabla_h^* v + \alpha(u,v)$$

We have (almost inadvertently) introduced a consistency error. Written as they are now, u does not satisfy the discrete equations.

Basic solvability and stability analysis

$$\|\nabla_h^* v\|_{0,\Omega}^2 \lesssim \|\nabla_h v\|_{0,\Omega}^2 + \alpha(v,v) =: \|v\|_h^2$$

The term $\alpha(v,v)$ penalizes discontinuities, but has some strange terms penalising that v doesn't satisfy the homog Dirichlet condition on Γ_0 and Ξ_-

Theorem

$$|B_h(u,v) - B_h(u^*,v)| \lesssim \|u - u^*\|_h \, \|v\|_h$$

$$B_h(u,u-v) - B_h(v,u-v) \gtrsim \|u - v\|_h^2$$

This implies unique solvability of

$$u_h \in V_h \qquad B(u_h,v_h) = \ell_h(v_h), \quad \forall v_h \in V_h$$

and

$$\|u_h\|_h \lesssim \sup \frac{|B(0,v_h)|}{\|v_h\|_h} + \sup \frac{|\ell_h(v_h)|}{\|v_h\|_h}$$

In our case, the bound includes the following terms:

$$\int_\Omega |f|^2 + \int_{\Gamma_0} \alpha |g_0|^2 + \int_\Xi \alpha |g_1|^2 + \int_\Xi \alpha |g_2|^2 + \int_{\Omega_-} |\mathbf{a}(\,\cdot\,,\mathbf{0})|^2 + \sup \frac{1}{\|\!| v_h \|\!|_h} \Big| \int_\Gamma \lambda v_h \Big|$$

CONNECTING BOTH SIDES

Two NtD solvers on opposite sides of Γ

$$
\begin{array}{ccccc}
\lambda & \xrightarrow{\text{LDG}(\lambda;\text{data})} & (u_h, \boldsymbol{\theta}_h, \boldsymbol{\sigma}_h) & \longrightarrow & u_h \text{ on } \Gamma \\
\shortparallel & & & & \not\shortparallel \\
\lambda & \xrightarrow{\text{BEM}(\lambda)} & (\varphi_h, \gamma_h) & \longrightarrow & \varphi_h(+\mathbb{P}_0)
\end{array}
$$

$u_h|_\Gamma$ and φ_h are not even in the same space
(one is discontinuous, the other one is continuous)

Discretization accomplished

New space: $X_h \subset L^2(\Gamma)$.

Fin $\lambda_h \in X_h^0$, compute

$$\begin{array}{ccccccc}
X_h^0 \ni \lambda_h & \xrightarrow{\text{LDG}(\lambda_h;\text{data})} & (u_h, \boldsymbol{\theta}_h, \boldsymbol{\sigma}_h) & \longrightarrow & u_h \text{ on } \Gamma \\
\shortparallel & & & & \neq \\
\lambda_h & \xrightarrow{\text{BEM}(\lambda_h)} & (\varphi_h, \gamma_h) & \longrightarrow & \varphi_h (+\mathbb{P}_0)
\end{array}$$

and impose

$$\int_\Gamma (\varphi_h - u_h)\, \xi_h = 0, \qquad \forall \xi_h \in X_h^0.$$

Very quickly, some comments

- There are three independent grids:

$(u_h, \theta_h, \sigma_h)$	λ_h	(φ_h, γ_h)
LDG grid	mortar grid	BEM grid

- The grids are independent *up to a point* (i.e., they are not!).
- The mortar space should not be too rich
- The mortar grid sees the other two, which are mutually invisible (see later).
- We can treat the implicit system

$$\int_\Gamma \left(NtD_h^{\text{ext}}(\lambda_h) - NtD_h^{\text{int}}(\lambda_h) \right) \xi_h = 0, \qquad \forall \xi_h \in X_h^0$$

and try to solve it...

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- There are three independent grids:

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$$\int_\Gamma \left(NtD_h^{\text{ext}}(\lambda_h) - NtD_h^{\text{int}}(\lambda_h)\right)\xi_h = 0, \qquad \forall \xi_h \in X_h^0$$

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and try to solve it...

...or unfold the system to obtain...

$$\begin{array}{llll} \text{LDGEqns}(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) & -T_h'\lambda_h & & = \text{data} \\ T_h u_h & & -R_h'\varphi_h & = 0 \\ & R_h\lambda_h & +\text{BEM\_Eqns}(\varphi_h, \gamma_h) & = 0 \end{array}$$

T_h:= mass matrix/operator trace(V_h) $\times X_h^0$
R_h:= mass matrix/operator $X_h^0 \times Y_h^0$

... and then think of iterations.

An idea for analysis

Compact form of the system

Find $u_h \in V_h, \lambda_h \in X_h^0$, s.t.

$$\begin{array}{lllll} B_h(u_h, v_h) & -\int_\Gamma \lambda_h v_h & = \text{data}, & \forall v_h \in V_h \\ \int_\Gamma u_h \xi_h & +\langle -NtD_h(\lambda_h), \xi_h \rangle & = 0, & \forall \xi_h \in X_h^0 \end{array}$$

The whole discrete operator is uniformly strongly monotone with respect to the norm (assuming it is a norm!)

$$|\!|\!| u |\!|\!|_h + |\lambda|_h$$

but uniform Lipschitz continuity requires new norms:

$$|\cdot|_h^{\text{new}} := |\cdot|_h + \|\alpha^{1/2}\cdot|_{0,\Gamma}$$

$$|\!|\!| \cdot |\!|\!|_h^{\text{new}} := |\!|\!| \cdot |\!|\!|_h + \varepsilon_h \| \cdot \|_{0,\Gamma}$$

where $\|\xi_h\|_{0,\Gamma} \lesssim \varepsilon_h |\xi_h|_h, \qquad \forall \xi_h \in X_h.$

*n*th adaptation of Céa–Strang estimates

$$
\begin{aligned}
&C_h(p_h, q_h) = \ell_h(q_h), \qquad \forall q_h \\
&C_h(p, p-q) - C_h(q, p-q) \gtrsim \|p-q\|_h^2 \\
&|C_h(p,q) - C_h(p^*, q)| \lesssim \|p - p^*\|_h^{\text{new}} \|q\|_h
\end{aligned}
$$

We have unique solvability and the estimate

$$
\|p - p_h\|_h \lesssim \inf \|p - q_h\|_h^{\text{new}} + \sup \frac{|C_h(p, r_h) - \ell_h(r_h)|}{\|r_h\|_h}
$$

With patience and a good hammer, we can make everything fit in our case. Some terms are delicate to bound.

The last idea...

Since λ, φ, γ are piecewise very smooth, can we take X_h, Y_h, Z_h very small?
Then we can reduce the system to

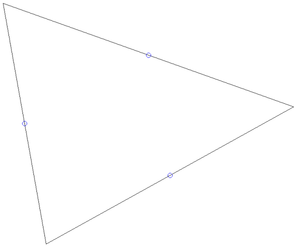
$$\text{LDG\_Eqns}(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) - T_h'(R_h' N t D_h R_h)^{-1} T_h u_h = \text{data}$$

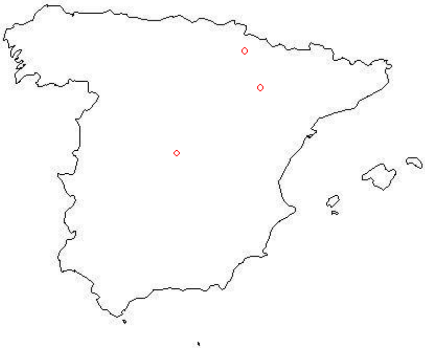
$T_h'(R_h' N t D_h R_h)^{-1} T_h \approx$ ABC restricted to the trace space of V_h. It'd be nice if we could have a smooth spherical/circular boundary and use spectral elements. But then you create two new problems: (a) You need isoparametric LDG. (b) You have to trick with the traces.

... as in Lenoir (95), Rapún & FJS (to appear)

New variational crime

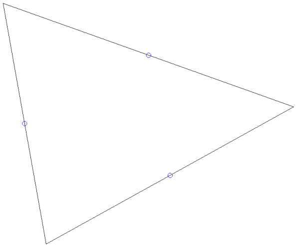
Blaming MC (after scattering)

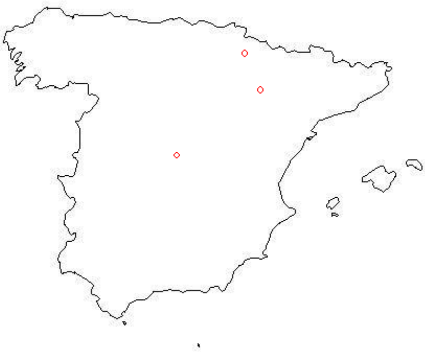




Coincidence?... I don't think so!

Blaming MC (after scattering)





Coincidence?... I don't think so!

And in a serious mood...

À Michel Crouzeix,

- avec admiration!
- plus d'admiration!!
- et encore plus d'admiration!!!

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