Numerical Method for Elliptic Multiscale Problems

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OUTLINE

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Introduction

Multiscale problems are described by partial differential equations with highly oscillatory coefficients.

$$\begin{cases} Lu = f\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \qquad \text{with} \qquad L = -\sum_{i,j=1}^{d} \frac{\partial}{\partial_j} \alpha_{ij} \frac{\partial}{\partial_i}.$$

Difficulty: accurate discrete solution of such problems requires a very fine discretisation.
$\Rightarrow$ High storage and computation costs.

Interest: the average behaviour of the elliptic oscillatory operator on a coarse scale taking into account the small scale features of the solution.

Different types of multiscale methods

1. Heterogeneous Multiscales Method [Weinan\Engquist,03]
2. Multiresolution Methods [Brewster\Beylkin, 95].
3. Generalised Finite Element Method: Babuška-Osborn,1d, [83]. Generalised to 2d by Hou-Wu [97].
4. Variational multiscale approach introduced by Hughes and Brezzi, Arbogast for a mixed variant.

Our approach: to provide a smoother elliptic operator which behaves like the original operator on a coarse mesh, with no smoothness or periodicity requirement.

1D Elliptic Problem

Let $\Omega = (0,1)$ and $V = H_0^1(\Omega)$. Let $L : V \to V'$ be the elliptic operator

$$L = -\frac{d}{dx}(\alpha(x)\frac{d}{dx})$$

where $\alpha \in L^\infty(\Omega)$, $\alpha(x) > 0$. Let u be the solution of the following problem

$$\begin{cases} L\,u = f & \text{in } \Omega \\ u(0) = u(1) = 0\,. \end{cases} \tag{1}$$

$\Rightarrow$ Oscillatory or jumping coefficients are allowed.

We need a small scale h to take care of the oscillatory coefficient.

Fine grid discretisation $\mathcal{T}_h$

The P^1-Lagrange FE space $V_h = \mathrm{span}\{b_1^h, \ldots, b_{N-1}^h\}$.

Let P_h be the following isomorphism

$$P_h : \mathbb{R}^{N-1} \longrightarrow V_h \subset V$$

$$v = (v_1, \ldots, v_{N-1}) \longmapsto P_h\, v = \sum_{i=1}^{N-1} v_i\, b_i^h\,.$$

Its adjoint is $R_h := P_h^* \in L(V', \mathbb{R}^{N-1})$.
The mass matrix: $M_h := R_h\, P_h \in \mathbb{R}^{(N-1)\times(N-1)}$.

The FE stiffness matrix L_h is given by

$$L_h = R_h\, L\, P_h.$$

Coarse grid discretisation $\mathcal{T}_H$, $H \gg h$

Let V_H be the linear FE space $V_H = \mathrm{span}\{b_1^H, \ldots, b_{M-1}^H\}$.
The grid $\mathcal{T}_H$ is nested in $\mathcal{T}_h$ in the sense that $V_H \subset V_h$.

Let P_H be the following isomorphism

$$P_H : \mathbb{R}^{M-1} \to V_H \subset V$$
$$v = (v_1, \ldots, v_{M-1}) \longmapsto P_H\, v = \sum_{i=1}^{M-1} v_i\, b_i^H\,.$$

Its adjoint is $R_H := P_H^* \in L(V', \mathbb{R}^{M-1})$.

The mass matrix: $M_H := R_H\, P_H \in \mathbb{R}^{(M-1)\times(M-1)}$.

Let $\|\cdot\|_2$ be the Euclidean norm and $\|\!|\cdot|\!\|$ the norm defined for a matrix $X \in \mathbb{R}^{(M-1)\times(M-1)}$ by

$$\|\!|X^{-1}|\!\| := \|P_H\, X^{-1} R_H\|_{L^2(\Omega)\leftarrow L^2(\Omega)} = \|M_H^{1/2} X^{-1} M_H^{1/2}\|_2\,.$$

Notation

The inclusion $V_H \subset V_h$ ensures that the following mappings are well defined:

1. the prolongation operator $P_{h \leftarrow H}$ from the coarse space V_H to the fine space V_h given by

$$P_{h \leftarrow H} = (P_h^{-1}\, P_H) : \mathbb{R}^{M-1} \longrightarrow \mathbb{R}^{N-1}\,.$$

2. the restriction operator $R_{H \leftarrow h} = P^*_{h \leftarrow H}$ from the fine space V_h to the coarse space V_H.

3. the normalised prolongation and restriction operators

$$\tilde{P}_{h \leftarrow H} = M_h\, P_{h \leftarrow H}\, M_H^{-1} \qquad \text{and} \qquad \tilde{R}_{H \leftarrow h} = \tilde{P}^*_{h \leftarrow H}.$$

Our approach

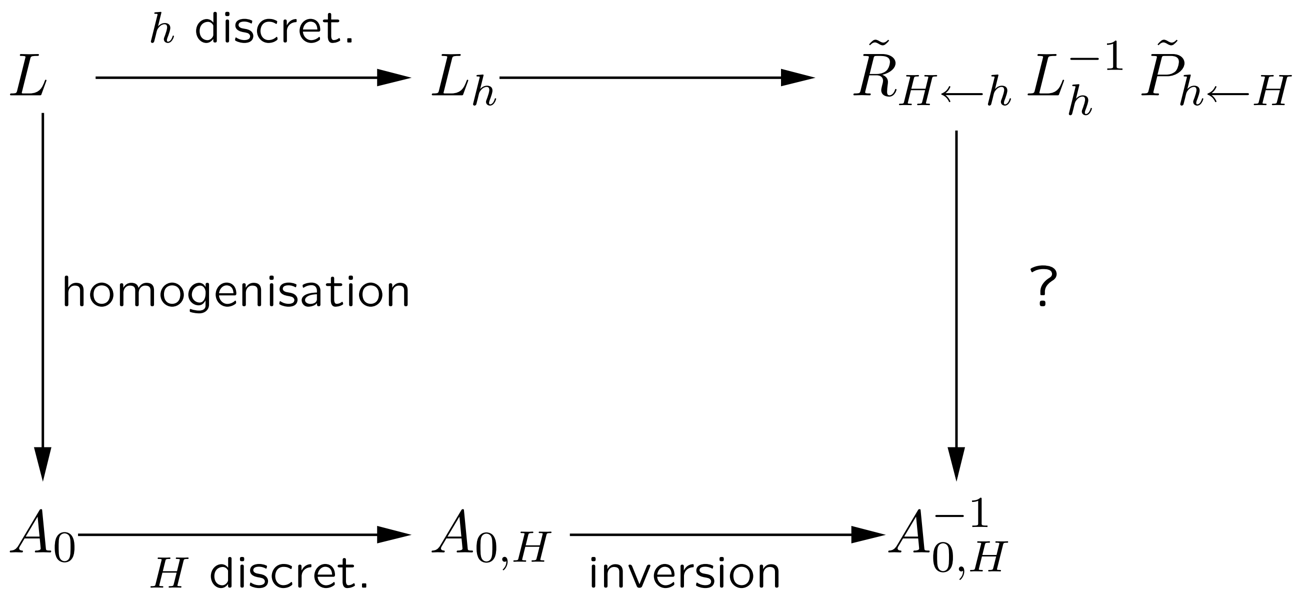
We consider the computation of $\mathcal{L}_H$

$$\mathcal{L}_H := \left(\tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H}\right)^{-1}$$

Question: Does it exist an elliptic operator $\mathcal{L}$, which discretisation is $\mathcal{L}_H$ on the coarse grid $\mathcal{T}_H$?

Remarks:

- No requirement of regularity or periodicity of the coefficient.
- Available in any dimensions.



We consider the following problem:

We are looking for an elliptic operator $A \in L(V, V')$ such that the inverse of its discretisation A_H^{-1} on the coarse grid satisfies:

$$A_H^{-1} \approx \tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H} \, .$$

Hope $A = -a\frac{d^2}{dx^2}$, a smoother than α.

Homogenisation

Let us consider the special case of a T-periodic coefficient α.

Homogenisation theory:
the exact solution $u = L^{-1}f$ is approximated by the homogenised one $u_0 = L_0^{-1}f$ with a precision depending on the period T.

The homogenised operator L_0 associated with L is defined by

$$L_0 = -\alpha_0 \frac{d^2}{dx^2} \qquad \text{with } \alpha_0 = \frac{1}{M(\frac{1}{\alpha})} \text{ and } M\left(\frac{1}{\alpha}\right) = \frac{1}{T}\int_T \frac{dx}{\alpha(x)}.$$

It is reasonable to expect that L_0 a good solution of our problem, i.e.:

$$L_{0,H}^{-1} \approx \tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H} \,.$$

Theoretical result

Let $L_{0,H}$ be the coarse FE-discretisation of L_0.

Comparison of $\tilde{R}_{H\leftarrow h}\, L_h^{-1}\, \tilde{P}_{h\leftarrow H}$ and $L_{0,H}^{-1}$ defined on the coarse level.

Theorem *The following error estimate holds*

$$\|\tilde{R}_{H\leftarrow h}\, L_h^{-1}\, \tilde{P}_{h\leftarrow H} - L_{0,H}^{-1}\| \;\leq\; C\left(\varepsilon(h) + T\, M\left(\frac{1}{\alpha}\right)(1+2T) + \varepsilon_0(H)\right),$$

where $\varepsilon(h)$ is a bound on the FE-discretisation error of L on the fine mesh $\mathcal{T}_h$
$\varepsilon_0(H)$ is the bound on the FE-discretisation error of L_0 on the coarse mesh $\mathcal{T}_H$.

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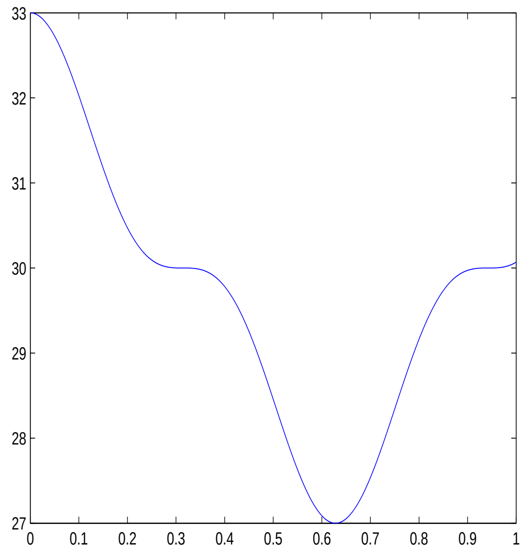
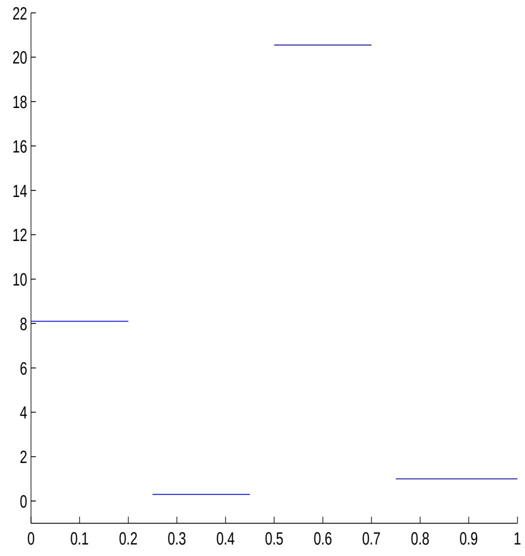
Numerical experiments

1. T-periodic coefficients

2. Non-periodic coefficients

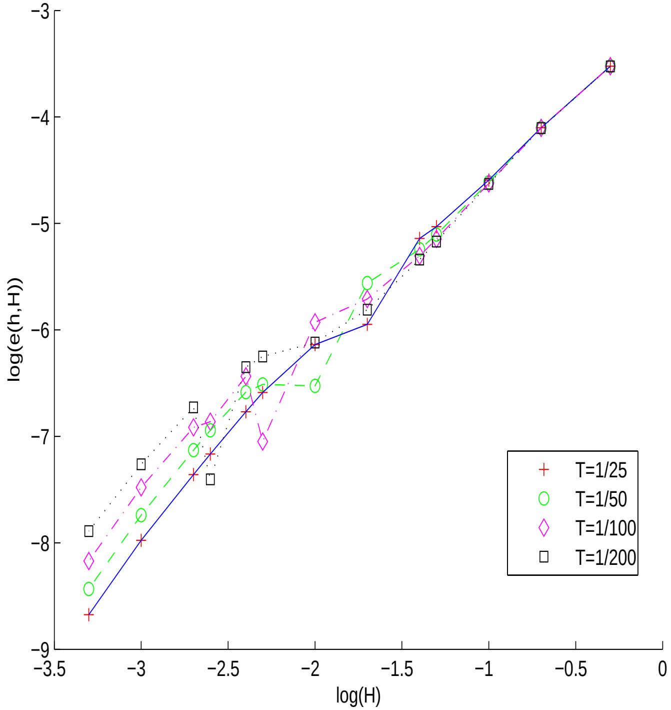




Smooth coefficient, small amplitude

Computation of the norm $\|\tilde{R}_{H\leftarrow h}\, L_h^{-1}\, \tilde{P}_{h\leftarrow H} - L_{0,H}^{-1}\|$ for:

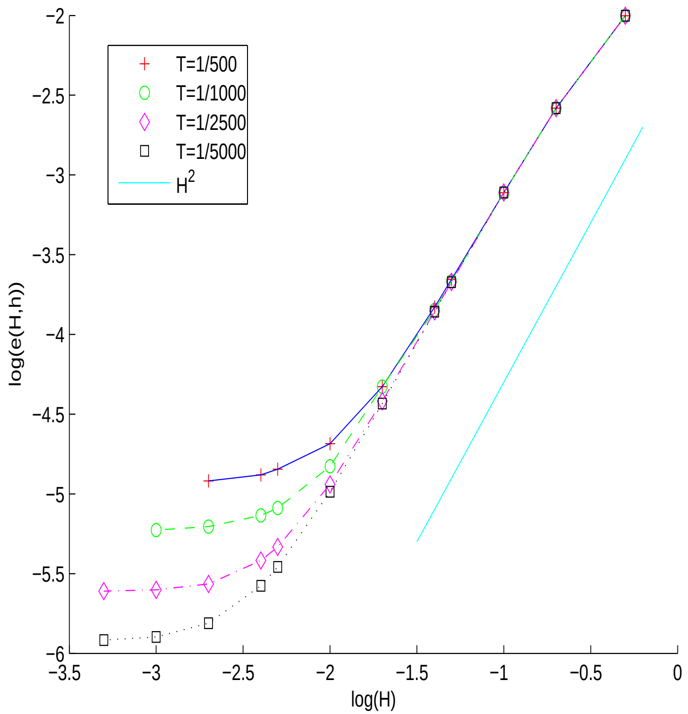
- small step size: h=1/4000
- large step size: H between 1/2 =2000h and 2h
- periods $T = 1/25,\ 1/50,\ 1/100,\ 1/200$.



The convergence rate of is of order 1.5 with a jump around $H = T/2$.

Piecewise constant coefficient

Computation of the norm $\|\tilde{R}_{H\leftarrow h}\, L_h^{-1}\, \tilde{P}_{h\leftarrow H} - L_{0,H}^{-1}\|$ for:
- small step size: $h = 10^{-5}$
- large step size: H between 1/2 and $50\,h$
- periods $T = 1/500,\ 1/1000,\ 1/2500,\ 1/5000.$



- plateau when T and H close to each other
- convergence of the norm when $T \to 0$

Numerical experiments

1. T-periodic coefficients

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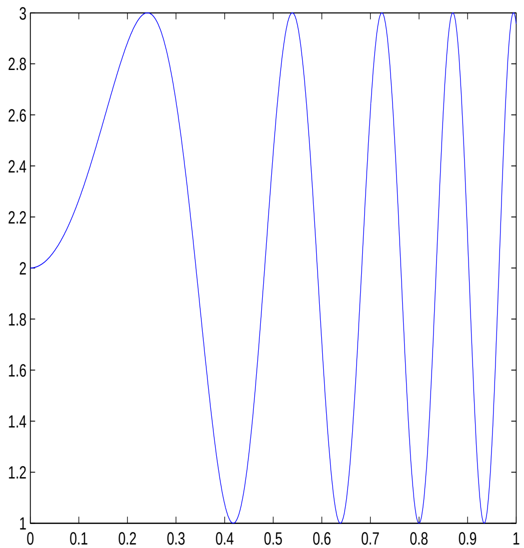
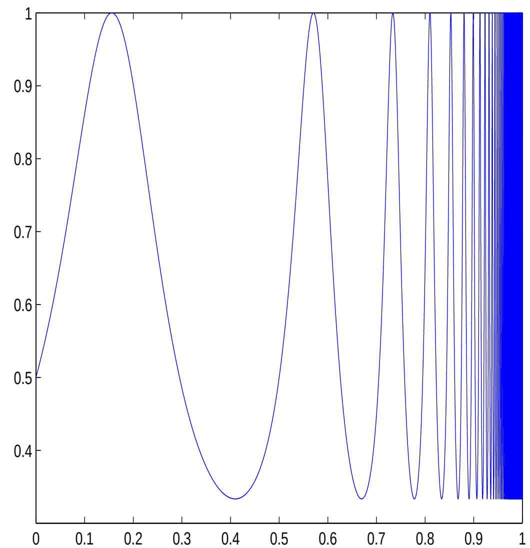
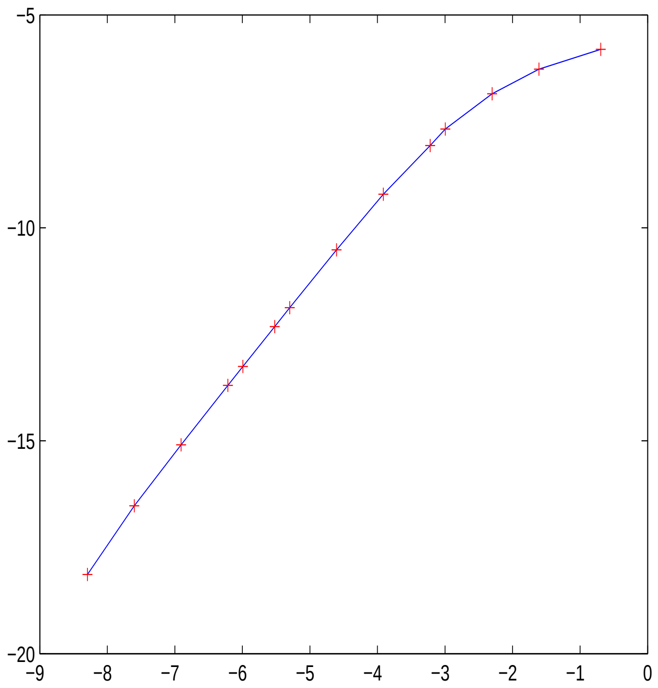




A smooth coefficient $\alpha(x) = 2 + \sin(27\,x^2)$

Computation of the norm $\|\tilde{R}_{H\leftarrow h}\, L_h^{-1}\, \tilde{P}_{h\leftarrow H} - L_{0,H}^{-1}\|$ for :

- the small step size h equals to $h = 1/1000$, $1/2000$, $1/4000$, $1/8000$,
- the coarse step size H ranges over $1/2$ to $1/4000$.



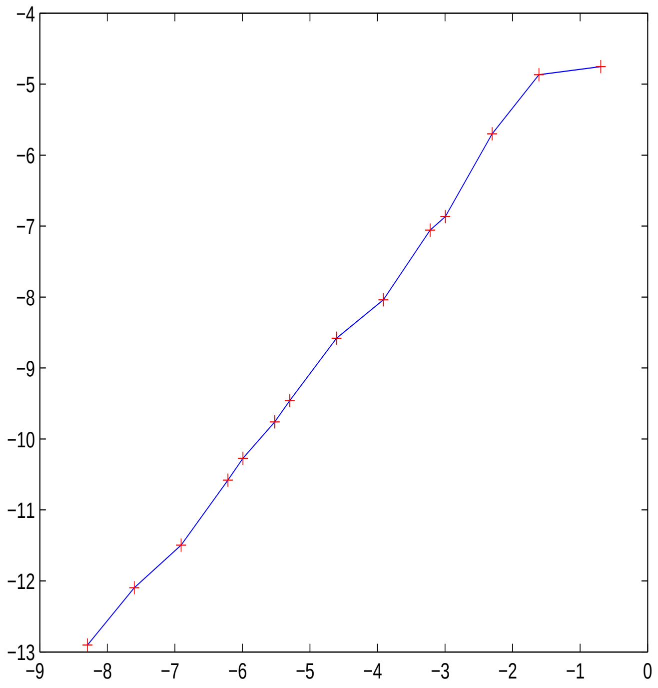
- Independence on the choice of the step size h.
- Good convergence almost order $\mathcal{O}(H^2)$ for $H < 1/25$.

$\Rightarrow$ Efficiency on coarse meshes.

An oscillating coefficient

This function contains a continuum of scales.

Computation of the norm $\|\tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H} - L_{0,H}^{-1}\|$ for :
- the small step size h equals to $h = 1/1000$, $1/2000$, $1/4000$, $1/8000$,
- the coarse step size H ranges over $1/2$ to $1/4000$.



- Dependence on the choice of the step size h is small.
- Good convergence almost order $\mathcal{O}(H)$.

Conclusion

- The choice of the operator A_0 defined by $A_0 = -a_0 \frac{d^2}{dx^2}$ satisfies
$$A_{0,H}^{-1} \approx \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} \, .$$
- The method is efficient and robust.
- No smoothness or periodicity requirement of the coefficient α.

Future work

- Generalisation to 2D.
- Difficulty since the Green function is not analytically available.