Numerical Method for Elliptic Multiscale Problems

Isabelle GREFF

Université de Pau et des Pays de l’Adour

Joint work with Professor W. Hackbusch, MPI Leipzig

CANUM 2006
OUTLINE

1. Introduction

2. Multiscale approach

3. Theory for periodic problems

4. Numerical Experiment
Introduction

Multiscale problems are described by partial differential equations with highly oscillatory coefficients.

\[
\begin{aligned}
L u &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

with

\[
L = - \sum_{i,j=1}^{d} \frac{\partial}{\partial j} \alpha_{ij} \frac{\partial}{\partial i}.
\]

**Difficulty:** accurate discrete solution of such problems requires a very fine discretisation.

⇒ High storage and computation costs.

**Interest:** the average behaviour of the elliptic oscillatory operator on a coarse scale taking into account the small scale features of the solution.
Different types of multiscale methods

1. Heterogeneous Multiscales Method [Weinan\Engquist,03]
2. Multiresolution Methods [Brewster\Beylkin, 95].
4. Variational multiscale approach introduced by Hughes and Brezzi, Arbogast for a mixed variant.

Our approach: to provide a smoother elliptic operator which behaves like the original operator on a coarse mesh, with no smoothness or periodicity requirement.
1D Elliptic Problem

Let $\Omega = (0, 1)$ and $V = H^1_0(\Omega)$. Let $L : V \to V'$ be the elliptic operator

$$L = -\frac{d}{dx}(\alpha(x)\frac{d}{dx})$$

where $\alpha \in L^\infty(\Omega)$, $\alpha(x) > 0$. Let $u$ be the solution of the following problem

$$
\begin{cases}
L u = f & \text{in } \Omega \\
u(0) = u(1) = 0.
\end{cases}
$$

$\Rightarrow$ Oscillatory or jumping coefficients are allowed.

We need a small scale $h$ to take care of the oscillatory coefficient.
Fine grid discretisation $\mathcal{T}_h$

The $P^1$-Lagrange FE space $V_h = \text{span}\{b^h_1, \ldots, b^h_{N-1}\}$.

Let $P_h$ be the following isomorphism

$$P_h : \mathbb{R}^{N-1} \rightarrow V_h \subset V$$

$$v = (v_1, \ldots, v_{N-1}) \mapsto P_h v = \sum_{i=1}^{N-1} v_i b^h_i.$$ 

Its adjoint is $R_h := P_h^* \in L(V', \mathbb{R}^{N-1})$.

The mass matrix: $M_h := R_h P_h \in \mathbb{R}^{(N-1) \times (N-1)}$.

The FE stiffness matrix $L_h$ is given by

$$L_h = R_h L P_h.$$
Coarse grid discretisation $\mathcal{T}_H$, $H \gg h$

Let $V_H$ be the linear FE space $V_H = \text{span}\{b_1^H, \ldots, b_{M-1}^H\}$. The grid $\mathcal{T}_H$ is nested in $\mathcal{T}_h$ in the sense that $V_H \subset V_h$.

Let $P_H$ be the following isomorphism

$$P_H : \mathbb{R}^{M-1} \rightarrow V_H \subset V$$

$$v = (v_1, \ldots, v_{M-1}) \mapsto P_H v = \sum_{i=1}^{M-1} v_i b_i^H.$$ 

Its adjoint is $R_H := P_H^* \in L(V', \mathbb{R}^{M-1})$.

The mass matrix: $M_H := R_H P_H \in \mathbb{R}^{(M-1) \times (M-1)}$.

Let $\| \cdot \|_2$ be the Euclidean norm and $\| \cdot \|$ the norm defined for a matrix $X \in \mathbb{R}^{(M-1) \times (M-1)}$ by

$$\| X^{-1} \| := \| P_H X^{-1} R_H \|_{L^2(\Omega) \rightarrow L^2(\Omega)} = \| M_H^{1/2} X^{-1} M_H^{1/2} \|_2.$$
Notation

The inclusion $V_H \subset V_h$ ensures that the following mappings are well defined:

1. the prolongation operator $P_{h\leftarrow H}$ from the coarse space $V_H$ to the fine space $V_h$ given by

   $$P_{h\leftarrow H} = (P_h^{-1} P_H) : \mathbb{R}^{M-1} \rightarrow \mathbb{R}^{N-1}.$$ 

2. the restriction operator $R_{H\leftarrow h} = P_{h\leftarrow H}^*$ from the fine space $V_h$ to the coarse space $V_H$.

3. the normalised prolongation and restriction operators

   $$\tilde{P}_{h\leftarrow H} = M_h P_{h\leftarrow H} M_H^{-1} \quad \text{and} \quad \tilde{R}_{H\leftarrow h} = \tilde{P}_{h\leftarrow H}^*.$$
1. Introduction

2. Multiscale approach

3. Theory for periodic problems

4. Numerical Experiment
Our approach

We consider the computation of $\mathcal{L}_H$

$$\mathcal{L}_H := \left( \tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H} \right)^{-1}$$

**Question:** Does it exist an elliptic operator $\mathcal{L}$, which discretisation is $\mathcal{L}_H$ on the coarse grid $T_H$?

**Remarks:**
- No requirement of regularity or periodicity of the coefficient.
- Available in any dimensions.
We consider the following problem:

We are looking for an elliptic operator $A \in L(V, V')$ such that the inverse of its discretisation $A_{H}^{-1}$ on the coarse grid satisfies:

$$A_{H}^{-1} \approx \tilde{R}_{H \leftarrow h} L_{h}^{-1} \tilde{P}_{h \leftarrow H}.$$ 

Hope $A = -a \frac{d^2}{dx^2}$, a smoother than $\alpha$. 

$$\tilde{R}_{H \leftarrow h} L_{h}^{-1} \tilde{P}_{h \leftarrow H}.$$
1. Introduction

2. Multiscale approach

3. Theory for periodic problems

4. Numerical Experiment
Homogenisation

Let us consider the special case of a $T$-periodic coefficient $\alpha$.

**Homogenisation theory:**

the exact solution $u = L^{-1}f$ is approximated by the homogenised one $u_0 = L_0^{-1}f$ with a precision depending on the period $T$.

The homogenised operator $L_0$ associated with $L$ is defined by

$$L_0 = -\alpha_0 \frac{d^2}{dx^2} \quad \text{with} \quad \alpha_0 = \frac{1}{M\left(\frac{1}{\alpha}\right)} \quad \text{and} \quad M\left(\frac{1}{\alpha}\right) = \frac{1}{T} \int_T \frac{dx}{\alpha(x)}.$$ 

It is reasonable to expect that $L_0$ a good solution of our problem, i.e.:

$$L_{0,H}^{-1} \approx \tilde{R}_H^{-1}L_h^{-1}\tilde{P}_h^{-1}H.$$
Theoretical result

Let $L_{0,H}$ be the coarse FE-discretisation of $L_0$.

Comparison of $\tilde{R}_{H\leftarrow h} L^{-1}_h \tilde{P}_{h\leftarrow H}$ and $L^{-1}_{0,H}$ defined on the coarse level.

**Theorem** The following error estimate holds

$$\| \tilde{R}_{H\leftarrow h} L^{-1}_h \tilde{P}_{h\leftarrow H} - L^{-1}_{0,H} \| \leq C \left( \varepsilon(h) + TM \left( \frac{1}{\alpha} \right) (1 + 2T) + \varepsilon_0(H) \right),$$

where $\varepsilon(h)$ is a bound on the FE-discretisation error of $L$ on the fine mesh $T_h$,

$\varepsilon_0(H)$ is the bound on the FE-discretisation error of $L_0$ on the coarse mesh $T_H$. 
1. Introduction

2. Multiscale approach

3. Theory for periodic problems

4. Numerical Experiment
Numerical experiments

1. $T$-periodic coefficients

2. Non-periodic coefficients
Smooth coefficient, small amplitude

Computation of the norm \[ \| \tilde{R}_{H-h} L_h^{-1} \tilde{P}_{h-H} - L_{0,H}^{-1} \| \] for:
- small step size: \( h=1/4000 \)
- large step size: \( H \) between \( 1/2 = 2000h \) and \( 2h \)
- periods \( T = 1/25, 1/50, 1/100, 1/200 \).

The convergence rate of is of order 1.5 with a jump around \( H = T/2 \).
Piecewise constant coefficient

Computation of the norm $\| \tilde{R}_{H-h} L_h^{-1} \tilde{P}_{h-H} - L_{0,H}^{-1} \|$ for:
- small step size: $h = 10^{-5}$
- large step size: H between $1/2$ and $50 \ h$
- periods $T = 1/500, 1/1000, 1/2500, 1/5000$

- plateau when $T$ and $H$ close to each other
- convergence of the norm when $T \to 0$
Numerical experiments

1. $T$-periodic coefficients

2. Non-periodic coefficients
A smooth coefficient $\alpha(x) = 2 + \sin(27 x^2)$

Computation of the norm $\| \tilde{R}_{h-H} L_h^{-1} \tilde{P}_{h-H} - L_{0,H}^{-1} \|$ for:
- the small step size $h$ equals to $h = 1/1000, 1/2000, 1/4000, 1/8000$,
- the coarse step size $H$ ranges over $1/2$ to $1/4000$.

- Independence on the choice of the step size $h$.
- Good convergence almost order $O(H^2)$ for $H < 1/25$.

$\Rightarrow$ Efficiency on coarse meshes.
An oscillating coefficient

This function contains a continuum of scales.

Computation of the norm \( \| \tilde{R}_{H-h} L_h^{-1} \tilde{P}_{h-H} - L_{0,H}^{-1} \| \) for:
- the small step size \( h \) equals to \( h = 1/1000, 1/2000, 1/4000, 1/8000 \),
- the coarse step size \( H \) ranges over \( 1/2 \) to \( 1/4000 \).

- Dependence on the choice of the step size \( h \) is small.
- Good convergence almost order \( \mathcal{O}(H) \).
Conclusion

• The choice of the operator $A_0$ defined by $A_0 = -a_0 \frac{d^2}{dx^2}$ satisfies

$$A_{0,H}^{-1} \approx \tilde{R}_H \leftarrow h L_h^{-1} \tilde{P}_h \leftarrow H.$$  

• The method is efficient and robust.

• No smoothness or periodicity requirement of the coefficient $\alpha$.

Future work

• Generalisation to 2D.

• Difficulty since the Green function is not analytically available.