

Higher Order Entropies

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Higher order entropies (1)

- **A Thermodynamic interpretation of Bernstein Method**

Consider the heat equation

$$\partial_t u - \Delta u = 0$$

Defining

$$\zeta = |\partial_x u|^2 = \partial_x u \cdot \partial_x u$$

we have

$$\partial_t \zeta - \Delta \zeta + 2|\partial_x^2 u|^2 = 0$$

where $|\partial_x^2 u|^2 = \partial_x^2 u : \partial_x^2 u$ and ζ plays the role of an entropy even though we also have

$$\partial_t u^2 - \Delta u^2 + 2|\partial_x u|^2 = 0$$

For more general equations

$$\zeta = u^2 + c_1 |\partial_x u|^2$$

Higher order entropies (2)

- Enskog second order expansion of kinetic entropy

The distribution function $f(t, x, c)$ of a monatomic gas is governed by Boltzmann equation

The kinetic entropy S^{kin} obeys the H theorem

$$S^{\text{kin}} = -k_{\text{B}} \int_{\mathbb{R}^n} f(\log f - 1) dc$$

Enskog expansion of f

$$f = f^{(0)}(1 + \varepsilon\phi^{(1)} + \mathcal{O}(\varepsilon^2))$$

$$\begin{cases} f^{(0)} & \text{is the Maxwellian distribution} \\ \phi^{(1)} & \text{is the perturbation associated with the Navier-Stokes regime} \end{cases}$$

Expansion of kinetic entropy S^{kin}

$$S^{\text{kin}} = S^{(0)} + \varepsilon^2 S^{(2)} + \mathcal{O}(\varepsilon^3)$$

Higher order entropies (3)

- Enskog second order kinetic entropy corrector

Fluid entropy $S^{(0)}$

$$S^{(0)} = -k_B \int_{\mathbb{R}^n} f^{(0)} (\log f^{(0)} - 1) dc$$

Enskog second order corrector $S^{(2)}$

$$S^{(2)} = -\frac{k_B}{2} \int_{\mathbb{R}^n} f^{(0)} (\phi^{(1)})^2 dc$$

Explicit evaluation

$$S^{(2)} = -\frac{1}{\rho} \left(\bar{\lambda} |\partial_x T|^2 + \frac{1}{2} \bar{\eta} |d|^2 \right)$$

where

$$d = \partial_x v + \partial_x v^t - \frac{2}{n} (\partial_x \cdot v) I$$
$$\bar{\lambda} = (1/2rc_p)\lambda^2/T^3 \quad \bar{\eta} = (1/4r)\eta^2/T^2$$

λ is the thermal conductivity, η the shear viscosity, and $c_p - c_v = r$

Higher order entropies (4)

- Enskog higher order expansion of kinetic entropy

Higher order expansion of f

$$f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \dots + \varepsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$$

Higher order expansion of S^{kin}

$$S^{\text{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \dots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$$

Structure of $S^{(2k)}$ in the absence of external forces

$$S^{(2k)} = r\rho \left(\frac{\eta}{\rho\sqrt{rT}} \right)^{2k} \sum_{\nu} c_{\nu} \prod_{1 \leq |\alpha| \leq 2k} \left(\frac{\partial^{\alpha} T}{T} \right)^{\nu_{\alpha}} \left(\frac{\partial^{\alpha} \rho}{\rho} \right)^{\nu'_{\alpha}} \left(\frac{\partial^{\alpha} v}{\sqrt{rT}} \right)^{\nu''_{\alpha}}$$

where η is the shear viscosity, $\nu = (\nu_{\alpha}, \nu'_{\alpha}, \nu''_{\alpha})_{1 \leq |\alpha| \leq 2k}$, and

$$\sum_{1 \leq |\alpha| \leq 2k} |\alpha| (\nu_{\alpha} + \nu'_{\alpha} + \nu''_{\alpha}) = 2k$$

The coefficients $c_{\alpha\beta\delta}$ are smooth bounded functions of $\log T$

Higher order entropies (5)

- **Kinetic entropy deviation estimators**

Integration by parts in $\int_{\mathbb{R}^n} S^{(2k)} dx$ to eliminate derivatives of order strictly greater than k

Use of classical interpolation inequalities

$$\left| \int_{\mathbb{R}^n} S^{(2k)} dx \right| \leq C \int_{\mathbb{R}^n} \gamma^{[k]} dx$$

$\gamma^{[k]}$ is a $(2k)^{\text{th}}$ order kinetic entropy deviation estimator

$$\gamma^{[k]} = r\rho \left(\frac{\eta}{\rho\sqrt{rT}} \right)^{2k} \left(\left| \frac{\partial^k T}{T} \right|^2 + \left| \frac{\partial^k v}{\sqrt{rT}} \right|^2 + \left| \frac{\partial^k \rho}{\rho} \right|^2 \right)$$

Other $(2k)^{\text{th}}$ order kinetic entropy deviation estimator

$$\tilde{\gamma}^{[k]} = r\rho \left(\frac{\eta}{\rho\sqrt{rT}} \right)^{2k} \left(|\partial^k \log T|^2 + |\partial^k (v/\sqrt{rT})|^2 + |\partial^k \log \rho|^2 \right)$$

Higher order entropies (6)

- **Kinetic entropy deviation estimators**

Zeroth order estimator

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = \frac{E^{(0)}}{T_\infty} - S^{(0)}$$

Kinetic entropy estimators of $(2k)^{\text{th}}$ order

$$\Gamma^{[k]} = \gamma^{[0]} + \dots + \gamma^{[k]}$$

$$\tilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \dots + \tilde{\gamma}^{[k]}$$

- **Parallel with scalar equation**

$$\zeta^{[k]} = u^2 + c_1 |\partial^1 u|^2 + \dots + c_k |\partial^k u|^2$$

Higher order entropies (7)

- **Enskog expansion of Fisher Information**

Leads to the same higher order estimators

- **Natural temperature scaling**

- **Persistence of kinetic entropy**

Higher order entropies are tools to estimate the solution of a fluid model

Different point of view from extended thermodynamics or burnett type models

Will require small Mach numbers, which is compatible with Enskog expansion from

$$\text{Ma} = \text{Kn Re}$$

where

$$\left\{ \begin{array}{ll} \text{Ma} = v/\bar{c} & \text{is the Mach number} \\ \text{Kn} = l/L & \text{is the Knudsen number} \\ \text{Re} = \rho v L/\eta & \text{is the Reynolds number} \end{array} \right.$$

Preliminary analysis (1)

- **Incompressible model**

$$\partial_x \cdot v = 0$$

$$\partial_t(\rho v) + \partial_x \cdot (\rho v \otimes v + pI) - \partial_x \cdot (\eta(T) d) = 0$$

$$\partial_t(\rho c_v T) + \partial_x \cdot (\rho c_v T v) - \partial_x \cdot (\lambda(T) \partial_x T) = \frac{1}{2} \eta(T) d : d$$

- **General assumptions**

$$v, T - T_\infty \in C([0, \bar{t}], H^l) \cap C^1([0, \bar{t}], H^{l-2}) \cap L^2([0, \bar{t}], H^{l+1}), \quad l \geq [n/2] + 2$$

- **Momentum equation**

$$\partial_t(\rho v) = \mathbb{P}(\partial_x \cdot (-\rho v \otimes v + \eta(T) d)), \quad \mathbb{P} = \mathbb{I} + R \otimes R$$

$$R = (R_1, \dots, R_n)^t, \quad R_i = (-\Delta)^{-1/2} \partial_i, \quad 1 \leq i \leq n$$

- **Pressure**

$$p = \sum_{1 \leq i, j \leq n} R_i R_j (\rho v_i v_j - \eta d_{ij})$$

Preliminary analysis (2)

- Simplifying assumptions

$$\lambda = \text{Cte}, \quad \eta = \text{Cte}, \quad c_v = \text{Cte}$$

- Pressure

$$p = \sum_{1 \leq i, j \leq n} R_i R_j (\rho v_i v_j)$$

- Second order entropy corrector

Specialize general formulas to the incompressible case

$$\gamma = \frac{A_\lambda}{T^{1+a}} |\partial_x T|^2 + \frac{1}{2} \frac{A_\eta}{T^a} |d|^2$$

Positive parameters at our disposal

$$A_\lambda > 0, \quad A_\eta > 0, \quad a > 0$$

Note that v scales as \sqrt{T} in γ , and that γ and $\gamma^{[1]}$ have similar properties

We will assume that $0 < a \leq 1$ in the following

Preliminary analysis (3)

- Balance equation for γ

$$\partial_t \gamma + \partial_x \cdot (v\gamma) + \partial_x \cdot \varphi + \pi + \Sigma + \omega = 0$$

where

$$\begin{aligned} \pi &= \frac{2A_\lambda \lambda}{\rho c_v} \frac{|\partial_x^2 T|^2}{T^{1+a}} + \frac{(1+a)(2+a)A_\lambda \lambda}{\rho c_v} \frac{|\partial_x T|^4}{T^{3+a}} \\ &+ \left(\frac{(1+a)A_\lambda \eta}{2\rho c_v} + \frac{a(1+a)A_\eta \lambda}{2\rho c_v} \right) \frac{|d|^2 |\partial_x T|^2}{T^{2+a}} + \frac{aA_\eta \eta}{4\rho c_v} \frac{|d|^4}{T^{1+a}} + \frac{A_\eta \eta}{\rho} \frac{|\partial_x d|^2}{T^a} \\ \Sigma &= -\frac{4(1+a)A_\lambda \lambda}{\rho c_v} \frac{\partial_x^2 T : \partial_x T \otimes \partial_x T}{T^{2+a}} - \left(\frac{2A_\lambda \eta}{\rho c_v} + \frac{aA_\eta \lambda}{\rho c_v} + \frac{aA_\eta \eta}{\rho} \right) \frac{\partial_x d : d \otimes \partial_x T}{T^{1+a}} \\ \omega &= A_\lambda \frac{d : \partial_x T \otimes \partial_x T}{T^{1+a}} + 2\frac{A_\eta}{\rho} \frac{d : \partial_x^2 p}{T^a} + 2A_\eta \frac{d : (\partial_x v \cdot \partial_x v)}{T^a} \\ \varphi &= \frac{aA_\eta \lambda}{2\rho c_v} \frac{|d|^2 \partial_x T}{T^{1+a}} - \frac{A_\eta \eta}{\rho} \frac{d : \partial_x d}{T^a} + \frac{(1+a)A_\lambda \lambda}{\rho c_v} \frac{|\partial_x T|^2 \partial_x T}{T^{2+a}} - \frac{2A_\lambda \lambda}{\rho c_v} \frac{\partial_x^2 T \cdot \partial_x T}{T^{1+a}} \end{aligned}$$

For a, b vectors, c, d matrices, $a \otimes b$ is the matrix with elements $a_i b_j$, $1 \leq i, j \leq n$,

$c \otimes b$ is the third order tensor $c_{ij} b_k$, $1 \leq i, j, k \leq n$; $c:d$ is the quantity $\sum_{ij} c_{ij} d_{ij}$ and $|c|^2 = c:c$

For e and f third order tensors $e:f$ denotes the quantity $\sum_{ijk} e_{ijk} f_{ijk}$ and $|e|^2 = e:e$

Preliminary analysis (4)

- **Unconditional entropicity**

Integrating the γ balance equation we obtain that

$$\partial_t \int_{\mathbb{R}^n} \gamma dx + \int_{\mathbb{R}^n} (\pi + \Sigma) dx = - \int_{\mathbb{R}^n} \omega dx$$

We will say that unconditional entropicity holds if there exists $c > 0$ such that

$$\frac{1}{c} \int_{\mathbb{R}^n} \pi dx \leq \int_{\mathbb{R}^n} (\pi + \Sigma) dx \leq c \int_{\mathbb{R}^n} \pi dx$$

for any $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, with $T \geq T_{\min} > 0$ and in this situation

$$\partial_t \int_{\mathbb{R}^n} \gamma dx + \frac{1}{c} \int_{\mathbb{R}^n} \pi dx \leq \int_{\mathbb{R}^n} |\omega| dx$$

- **Proposition :** Assume that the parameter a associated with γ is such that

$$0 < a < \inf \left(\frac{4n-1}{2n^2+1}, 2 \left(\frac{\lambda}{\eta c_v} + \frac{\eta c_v}{\lambda} \right)^{-1} \right)$$

Then there exists positive constants A_λ and A_η such that unconditional entropicity holds. On the other hand, unconditional entropicity does not hold when a is close to unity.

Preliminary analysis (5)

- **Sketch of the Proof**

Explicit evaluation of

$$\int_{\mathbb{R}^n} (\pi + \Sigma) dx$$

Integration by parts and polar decomposition of the Hessian matrix $\partial_x^2 T$

$$\partial_x^2 T = \widehat{\partial_x^2 T} + (\Delta T/n)\mathbb{I}$$

Binomial formula

Counter exemple for $a = 1$

- **Inequalities**

One can establish that for $a \neq -2$, $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $T \geq T_{\min} > 0$

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}} dx \leq c \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}} dx$$

Different inequalities will be established with powers of $\|\log T\|_{BMO}$ in the right hand side

Preliminary analysis (6)

- **Expression of $\partial_x v$ in terms of d**

For any $v \in H^1$ and any index pair (i, j) we have

$$2\partial_j v_i = d_{ij} - \sum_{1 \leq l \leq n} R_l R_j d_{li} + \sum_{1 \leq l \leq n} R_l R_i d_{lj}$$

and we also have

$$2\partial_j \partial_k v_i = \partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}$$

- **Estimates of convective terms**

Assume that $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \geq T_{\min} > 0$, and $a \leq 1/3$. Then

$$\int_{\mathbb{R}^n} |\omega| dx \leq c \left(\int_{\mathbb{R}^n} \pi dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |d|^2 dx \right)^{1/2} \sup_{\mathbb{R}^n} T^{(1-a)/2}$$

We also have

$$\int_{\mathbb{R}^n} |\omega| dx \leq c \left(\int_{\mathbb{R}^n} \pi dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \gamma dx \right)^{1/2} \sup_{\mathbb{R}^n} T^{1/2}$$

Preliminary analysis (7)

- **Temperature weights**

Nonlinear convective terms introduce a multiplication by the velocity field v

Since v scales as \sqrt{T} we recover the \sqrt{T} weight of the convective terms

The $\sup_{\mathbb{R}^n} T$ factors cannot be controlled by $\int_{\mathbb{R}^n} \pi \, dx$ and $\int_{\mathbb{R}^n} \gamma \, dx$

Small values of a yields unconditional entropicity but prevent majorization of convective terms

Larger values of a yields majorization of convective terms but prevent unconditional entropicity

- **Partial conclusion**

Temperature dependence of transport coefficients λ and η

Conditional entropicity properties

- **Remarks**

Same conclusions for periodic domains

Assuming $\partial_x T/T \in L^2 \cap L^4$ when $n = 3$ implies that $\log T$ has a finite limit at infinity and $T_\infty > 0$

Weighted inequalities (0)

- **Differential identities**

Letting $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we have

$$\partial^\alpha (fg) = \sum_{0 \leq \beta \leq \alpha} c_{\alpha\beta} \partial^\beta f \partial^{\alpha-\beta} g$$

For f and g smooth scalar functions we have

$$\partial^\alpha (g \circ f) = \sum_{\sigma\mu} c_{\sigma\mu} \partial^\sigma g \prod_{1 \leq |\beta| \leq |\alpha|} (\partial^\beta f)^{\mu_\beta}$$

where $c_{\sigma\mu}$ are nonnegative integer coefficients

The sum is over $1 \leq \sigma \leq |\alpha|$, $\mu = (\mu_\beta)_{1 \leq |\beta| \leq |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, such that

$$\sum_{1 \leq |\beta| \leq |\alpha|} \mu_\beta = \sigma, \quad \sum_{1 \leq |\beta| \leq |\alpha|} \beta \mu_\beta = \alpha$$

so that we have in particular $\sum_\beta |\beta| \mu_\beta = |\alpha|$.

Weighted inequalities (1)

- **A_p class**

For $1 < p < \infty$, $g \in L^1_{loc}(\mathbb{R}^n)$, positive and locally integrable, g is in the A_p class if

$$[g]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q g \, dx \right) \left(\frac{1}{|Q|} \int_Q g^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty$$

We have $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithm in BMO

- **BMO space**

A locally summable function f belongs to the space $BMO(\mathbb{R}^n)$ when

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - \bar{f}_Q| \, dx < \infty$$

- **Fine properties of BMO**

There exists constants $b(n)$ and $B(n)$ such that for any $\theta \in \mathbb{R}$, $u \in BMO$, and $1 < p < \infty$

$$|\theta| \|u\|_{BMO} < b(n)/2, \quad |\theta| \|u\|_{BMO} < (p-1)b(n)/2$$

implies that $\exp(\theta u) \in A_p$ and $[\exp(\theta u)]_{A_p} \leq (1 + B(n))^p$

Weighted inequalities (2)

- **Weighted Calderón-Zygmund operators**

Let \mathcal{G} be a Calderón-Zygmund operator, $1 < p < \infty$, and g be a weight in A_p

Then \mathcal{G} is bounded in $L^p(gdx)$, or equivalently, $g^{1/p}\mathcal{G}g^{-1/p}$ is bounded in L^p with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p, [g]_{A_p})$

- **Calderón-Zygmund operators**

Let $\mathcal{G} : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator and K the restriction of the distribution kernel associated with \mathcal{G} to the open set $x \neq y$ of $\mathbb{R}^n \times \mathbb{R}^n$. We say that \mathcal{G} is a Calderón-Zygmund operator when

- (i) K is a locally integrable function and there exists c_0 such that for $x \neq y$ $|K(x, y)| \leq c_0|x - y|^{-n}$
- (ii) There exists $\delta \in (0, 1]$ and c_1 such that for $x \neq y$ and $|x' - x| \leq \frac{1}{2}|x - y|$ we have $|K(x', y) - K(x, y)| \leq c_1|x' - x|^\delta|x - y|^{-n-\delta}$
- (iii) Similarly if $x \neq y$ and $|y' - y| \leq \frac{1}{2}|x - y|$ we have $|K(x, y') - K(x, y)| \leq c_1|y' - y|^\delta|x - y|^{-n-\delta}$
- (iv) \mathcal{G} can be extended into a continuous linear operator over $L^2(\mathbb{R}^n)$ with norm lower or equal to c_2

Weighted inequalities (3)

- **Weighted multilinear estimates**

Let k, l be positive integers, and $\alpha^j, 1 \leq j \leq l$, be multiindices such that $|\alpha^j| \geq 1, 1 \leq j \leq l$, and $k = \sum_{1 \leq j \leq l} |\alpha^j|$. Let $1 < p < \infty, g \in A_p$, and u_1, \dots, u_l , be such that there exist constants $u_{j,\infty}$ with $u_j - u_{j,\infty} \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $g^{\frac{1}{p}} \partial^k u_j \in L^p, 1 \leq j \leq l$. There exists $c = c(k, n, p, [g]_{A_p})$ such that

$$\|g^{1/p} \prod_{1 \leq j \leq l} \partial^{\alpha^j} u_j\|_{L^p} \leq c \left(\sum_{1 \leq j \leq l} \|u_j\|_{BMO} \right)^{l-1} \left(\sum_{1 \leq j \leq l} \|g^{1/p} \partial^k u_j\|_{L^p} \right)$$

where we define

$$\|g^{1/p} \partial^k v\|_{L^p}^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\mathbb{R}^n} g |\partial^\alpha v|^p dx$$

- **Sketch of the Proof**

Fourier transform and density of $\mathcal{D}(\mathbb{R}^n)$ in $H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$

Coifman Meyer theory of multilinear operators

Continuity of weighted Calderón-Zygmund operators

- **Remark**

$\mathcal{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ if and only if $k \geq n/2$

Weighted inequalities (4)

- **Weighted multilinear estimates for τ and w**

Let $k \geq 1$, $\bar{\theta} > 0$, $1 < p < \infty$, τ such that $\tau - \tau_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ for some constant τ_∞ . There exist positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$ such that if $\|\tau\|_{BMO} < \delta$, then for any $|\theta| \leq \bar{\theta}$, any integer $l \geq 1$, and any multiindices α^j , $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1 \leq j \leq l} |\alpha^j| = k$, whenever $\exp(\theta\tau/p)\partial^k\tau \in L^p(\mathbb{R}^n)$, we have

$$\left\| e^{\frac{\theta\tau}{p}} \prod_{1 \leq j \leq l} (\partial^{\alpha^j} \tau) \right\|_{L^p} \leq c \|\tau\|_{BMO}^{l-1} \left\| e^{\frac{\theta\tau}{p}} \partial^k \tau \right\|_{L^p}.$$

Further assuming that $w \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $e^{\theta\tau/p}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \leq \bar{l} \leq l$, we have

$$\left\| e^{\frac{\theta\tau}{p}} \prod_{1 \leq j \leq \bar{l}} (\partial^{\alpha^j} \tau) \prod_{\bar{l}+1 \leq j \leq l} (\partial^{\alpha^j} w) \right\|_{L^p} \leq c \left(\|\tau\|_{BMO} + \|w\|_{BMO} \right)^{l-1} \left(\left\| e^{\frac{\theta\tau}{p}} \partial^k \tau \right\|_{L^p} + \left\| e^{\frac{\theta\tau}{p}} \partial^k w \right\|_{L^p} \right)$$

where, in the left hand side, we have denoted by w any of its components w_1, \dots, w_n

- **Sketch of the Proof**

Weighted multilinear estimates applied to τ and w and properties of BMO

Weighted inequalities (5)

- **Auxiliary functions**

$$\tau = \log T \quad w = \frac{v}{\sqrt{T}}$$

- **Derivatives of auxiliary functions**

For T smooth, positive and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we have

$$\frac{\partial^\alpha T}{T} = \partial^\alpha \tau + \sum_{\mu} c_\mu \prod_{1 \leq |\beta| \leq |\alpha| - 1} (\partial^\beta \tau)^{\mu_\beta}$$

where $\mu = (\mu_\beta)_{1 \leq |\beta| \leq |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, c_μ are nonnegative integer coefficients, and

$$\sum_{1 \leq |\beta| \leq |\alpha|} \beta \mu_\beta = \alpha$$

so that we have in particular $\sum_{\beta} |\beta| \mu_\beta = |\alpha|$. Similarly, we have for $1 \leq i \leq n$

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu \tilde{\alpha}} c_{\mu \tilde{\alpha}} \prod_{1 \leq |\beta| \leq |\alpha|} (\partial^\beta \tau)^{\mu_\beta} \partial^{\tilde{\alpha}} w_i + \sum_{\mu} c_{\mu 0} \prod_{1 \leq |\beta| \leq |\alpha|} (\partial^\beta \tau)^{\mu_\beta} w_i,$$

with $0 \leq \tilde{\alpha} \leq \alpha$ and $\sum_{1 \leq |\beta| \leq |\alpha|} \beta \mu_\beta + \tilde{\alpha} = \alpha$

Weighted inequalities (6)

- **Weighted multilinear estimates for T and v**

Let $k \geq 1$, $\bar{\theta} > 0$, $1 < p < \infty$, T with $T \geq T_{\min} > 0$ and $T - T_{\infty} \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ where $T_{\infty} > 0$. There exist positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$ such that if $\|\log T\|_{BMO} < \delta$, then for any $|\theta| \leq \bar{\theta}$, any integer $l \geq 1$, and any multiindices α^j , $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1 \leq j \leq l} |\alpha^j| = k$, whenever $T^{\theta/p}(\partial^k T)/T \in L^p(\mathbb{R}^n)$, we have

$$\left\| T^{\frac{\theta}{p}} \prod_{1 \leq j \leq l} \left(\frac{\partial^{\alpha^j} T}{T} \right) \right\|_{L^p} \leq c \|\log T\|_{BMO}^{l-1} \left\| T^{\frac{\theta}{p}} \frac{\partial^k T}{T} \right\|_{L^p}.$$

Further assuming $v \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$, whenever $T^{\theta/p}(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$, we have for $0 \leq \bar{l} \leq l$

$$\left\| T^{\frac{\theta}{p}} \prod_{1 \leq j \leq \bar{l}} \left(\frac{\partial^{\alpha^j} T}{T} \right) \prod_{\bar{l}+1 \leq j \leq l} \left(\frac{\partial^{\alpha^j} v}{\sqrt{T}} \right) \right\|_{L^p} \leq c \left(\|\log T\|_{BMO} + \left\| \frac{v}{\sqrt{T}} \right\|_{L^\infty} \right)^{l-1} \left(\left\| T^{\frac{\theta}{p}} \frac{\partial^k T}{T} \right\|_{L^p} + \left\| T^{\frac{\theta}{p}} \frac{\partial^k v}{\sqrt{T}} \right\|_{L^p} \right)$$

where, in the left hand member, we have denoted by v any of its components v_1, \dots, v_n .

- **Sketch of the Proof**

Weighted multilinear estimates are applied to τ and w and transported to T and v

Weighted inequalities (7)

- **Weighted interpolation inequality**

Let $1 < q < \infty, 1 < r < \infty, k \geq 1, 0 \leq j \leq k, g \in A_r \cap A_q = A_{\min(q,r)}$ and let p be such that

$$\frac{1}{p} = \frac{k-j}{k} \frac{1}{q} + \frac{j}{k} \frac{1}{r}$$

For $v \in L^q(gdx)$ and $\partial^k v \in L^r(gdx)$, we have $\partial^j v \in L^p(gdx)$ and there exists $\mathcal{C}(n, k, q, r, [g]_{A_q}, [g]_{A_r})$ with

$$\left(\int_{\mathbb{R}^n} g |\partial^j v|^p dx \right)^{\frac{1}{p}} \leq \mathcal{C} \left(\int_{\mathbb{R}^n} g |v|^q dx \right)^{\left(1 - \frac{j}{k}\right) \frac{1}{q}} \left(\int_{\mathbb{R}^n} g |\partial^k v|^r dx \right)^{\frac{j}{k} \frac{1}{r}}.$$

- **Weighted interpolation of derivatives**

Let $1 < r < \infty, k \geq 1, 1 \leq j \leq k, g \in A_r$ and $1/p = j/kr$. Then for v with $v - v_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $\partial^k v \in L^r(gdx)$, we have $\partial^j v \in L^p(gdx)$ and there exists $\mathcal{C}(n, k, r, [g]_{A_r})$ such that

$$\left(\int_{\mathbb{R}^n} g |\partial^j v|^{\frac{rk}{j}} dx \right)^{\frac{j}{rk}} \leq \mathcal{C} \|v\|_{BMO}^{1 - \frac{j}{k}} \left(\int_{\mathbb{R}^n} g |\partial^k v|^r dx \right)^{\frac{j}{rk}}$$

Weighted inequalities (8)

- **Sketch of the Proofs**

- (i) Induction on k and Grafakos and Torres theory of multilinear singular integral operators
- (ii) Letting $p = r$, $|\alpha^j| = 1$, and $k = l$ in the weighted estimates of Coifman Meyer type we obtain the inequality for $j = 1$. For $1 < j < k$ we can then interpolate $\partial^j v$ between $\partial^1 v$ and $\partial^k v$

- **Remark**

As a special case we obtain that for $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \geq T_{\min} > 0$ and $\|\log T\|_{BMO}$ small enough, we have

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}} dx \leq c \|\log T\|_{BMO}^2 \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}} dx$$

Higher order entropy estimates (1)

- **Incompressible model**

$$\partial_x \cdot v = 0$$

$$\partial_t(\rho v) + \partial_x \cdot (\rho v \otimes v + pI) - \partial_x \cdot (\eta(T) d) = 0$$

$$\partial_t(\rho c_v T) + \partial_x \cdot (\rho c_v T v) - \partial_x \cdot (\lambda(T) \partial_x T) = \frac{1}{2} \eta(T) d : d$$

- **General assumptions**

$$v, T - T_\infty \in C([0, \bar{t}], H^l) \cap C^1([0, \bar{t}], H^{l-2}) \cap L^2([0, \bar{t}], H^{l+1}), \quad l \geq [n/2] + 2$$

- **Momentum equation**

$$\partial_t(\rho v) = \mathbb{P}(\partial_x \cdot (-\rho v \otimes v + \eta(T) d)), \quad \mathbb{P} = \mathbb{I} + R \otimes R$$

$$R = (R_1, \dots, R_n)^t, \quad R_i = (-\Delta)^{-1/2} \partial_i, \quad 1 \leq i \leq n$$

- **Pressure**

$$p = \sum_{1 \leq i, j \leq n} R_i R_j (\rho v_i v_j - \eta d_{ij})$$

Higher order entropy estimates (2)

- **Temperature dependent transport coefficients**

There exist $\varkappa, \underline{a} > 0, \bar{a} > 0$, and $\bar{a}_\sigma > 0$ for any $\sigma \geq 1$ such that

$$\underline{a} T^\varkappa \leq \lambda/c_v \leq \bar{a} T^\varkappa, \quad \underline{a} T^\varkappa \leq \eta \leq \bar{a} T^\varkappa, \quad T^\sigma (|\partial_T^\sigma \lambda| + |\partial_T^\sigma \eta|) \leq \bar{a}_\sigma T^\varkappa$$

Kinetic theory suggests $1/2 \leq \varkappa \leq 1$ but $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate

We assume for the sake of simplicity that $c_v = \text{Cte}$

- **Interaction potentials between particles**

Power laws exactly valid for points centers of repulsion

Power laws asymptotically valid for Lenard-Jones interaction potentials

Assumptions also valid for polyatomic gases

- **Rescaling of solutions**

In the special case $\lambda = \alpha_\lambda T^\varkappa, \eta = \alpha_\eta T^\varkappa, c_v = \text{Cte}$, if $v(t, x)$ and $T(t, x)$ are a solution then

$$\xi v(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x), \quad \xi^2 T(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x)$$

are also a solution for any positive ξ

Higher order entropy estimates (3)

- **Higher order entropies**

Specialize general formulas to the incompressible case

$$\gamma^{[k]} = A_{\lambda}^{[k]} \frac{|\partial^k T|^2}{T^{1+a_k}} + A_{\eta}^{[k]} \frac{|\partial^k v|^2}{T^{a_k}}$$

$$|\partial^k T|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^{\alpha} T)^2, \quad |\partial^k v|^2 = \sum_{1 \leq i \leq n} |\partial^k v_i|^2$$

where $A_{\lambda}^{[k]} > 0$, $A_{\eta}^{[k]} > 0$, and $a_k \in \mathbb{R}$

- **Modified higher order entropies**

Denoting $\tau = \log T$ and $w = v/\sqrt{T}$ we define

$$\tilde{\gamma}^{[k]} = \exp((1 - a_k)\tau) \left(A_{\lambda}^{[k]} |\partial^k \tau|^2 + A_{\eta}^{[k]} |\partial^k w|^2 \right)$$

The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ will be shown to have similar properties

Higher order entropy estimates (4)

- **Zeroth order modified entropy**

We define $\gamma^{[0]} = \tilde{\gamma}^{[0]}$, for $0 < a_0 \leq 1$, by

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = (A_\lambda^{[0]} + A_\eta^{[0]})\zeta^{[0]}$$

where $A_\lambda^{[0]} > 0$, $A_\eta^{[0]} > 0$, and

$$\zeta^{[0]} = \begin{cases} \frac{T - T_\infty}{T_\infty} - \log\left(\frac{T}{T_\infty}\right) + \frac{1}{2} \frac{v^2}{c_v T_\infty}, & \text{if } a_0 = 1 \\ \frac{T - T_\infty}{T_\infty^{a_0}} - \frac{T^{1-a_0} - T_\infty^{1-a_0}}{1 - a_0} + \frac{1}{2} \frac{v^2}{c_v T_\infty^{a_0}}, & \text{if } 0 < a_0 < 1 \end{cases}$$

- **Higher order entropy estimators**

We define the $(2k)^{\text{th}}$ order kinetic entropy estimators by

$$\Gamma^{[k]} = \gamma^{[0]} + \dots + \gamma^{[k]} \quad k \geq 0$$

$$\tilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \dots + \tilde{\gamma}^{[k]} \quad k \geq 0$$

Higher order entropy estimates (5)

- **Governing equation for $\gamma^{[k]}$**

For $k \geq 1$ and (v, T) smooth solution of the incompressible Navier-Stokes equations we have

$$\partial_t \gamma^{[k]} + \partial_x \cdot (v \gamma^{[k]}) + \partial_x \cdot \varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0$$

where

$$\pi_\gamma^{[k]} = \frac{2\lambda A_\lambda^{[k]}}{\rho c_v} \frac{|\partial^{k+1} T|^2}{T^{1+a_k}} + \frac{2\eta A_\eta^{[k]}}{\rho} \frac{|\partial^{k+1} v|^2}{T^{a_k}}$$

$$\Sigma_\gamma^{[k]} = \sum_{\sigma \nu \mu} T^{\sigma-\varkappa} (c_{\sigma \nu \mu} \partial_T^\sigma \lambda + c'_{\sigma \nu \mu} \partial_T^\sigma \eta) \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma \nu \mu \mathcal{R}} c_{\sigma \nu \mu \mathcal{R}} \Pi_\nu^{(k+1)} \mathcal{R} (T^{\sigma-\varkappa} \partial_T^\sigma \eta \Pi_\mu^{(k+1)})$$

where $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1 \leq |\alpha| \leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1 \leq |\alpha| \leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and \mathcal{R} singular operator in the form $T^{-\theta} R_i R_j T^\theta$ with $\theta = (a_k + \varkappa)/2$, and

$$\Pi_\nu^{(k+1)} = T^{(1-a_k+\varkappa)/2} \prod_{1 \leq |\alpha| \leq k+1} \left(\frac{\partial^\alpha T}{T} \right)^{\nu_\alpha} \prod_{1 \leq |\alpha| \leq k+1} \left(\frac{\partial^\alpha v}{\sqrt{T}} \right)^{\nu'_\alpha}$$

where v denotes any of its components v_1, \dots, v_n , and μ and ν must be such that

$$\sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\nu_\alpha + \nu'_\alpha) = \sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\mu_\alpha + \mu'_\alpha) = k+1, \quad \sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leq 1$$

so that there is at most one derivative of $(k+1)^{\text{th}}$ order in the product $\Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$.

Higher order entropy estimates (6)

- **Governing equation for $\gamma^{[k]}$ (continued)**

The term $\omega_\gamma^{[k]}$ is given by

$$\omega_\gamma^{[k]} T^{-(1-2\kappa+a_{k-1}-a_k)/2} = \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\nu\mu\mathcal{R}} c_{\nu\mu\mathcal{R}} \Pi_\nu^{(k)} \mathcal{R}(\Pi_\mu^{(k+1)})$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1 \leq |\alpha| \leq k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k, \quad \sum_{1 \leq |\alpha| \leq k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k + 1$$

so that in particular $\sum_{|\alpha|=k+1} (\mu_\alpha + \mu'_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$ and the singular operator \mathcal{R} is in the form $T^{-\theta} R_i R_j T^\theta$ with $\theta = (1 + a_k - \kappa)/2$ and $1 \leq i, j \leq n$. Finally the flux $\varphi_\gamma^{[k]} = (\varphi_{\gamma 1}^{[k]}, \dots, \varphi_{\gamma n}^{[k]})$ is in the form

$$\varphi_{\gamma l}^{[k]} T^{-\frac{a_{k-1}-a_k}{2}} = \sum_{\sigma\nu\mu l} T^{\sigma-\kappa} (c_{\sigma\nu\mu l} \partial_T^\sigma \lambda + c'_{\sigma\nu\mu l} \partial_T^\sigma \eta) \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu\mathcal{R}l} c_{\sigma\nu\mu\mathcal{R}l} \Pi_\nu^{(k)} \mathcal{R}(T^{\sigma-\kappa} \partial_T^\sigma \eta \Pi_\mu^{(k+1)})$$

Higher order entropy estimates (7)

- **Governing equation for $\tilde{\gamma}^{[k]}$**

For $k \geq 1$ and (v, T) smooth solution of the incompressible Navier-Stokes equations we have

$$\partial_t \tilde{\gamma}^{[k]} + \partial_x \cdot (v \tilde{\gamma}^{[k]}) + \partial_x \cdot \varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0$$

where

$$\pi_{\tilde{\gamma}}^{[k]} = e^{(1-a_k)\tau} \left(\frac{2\lambda A_\lambda^{[k]}}{\rho c_v} |\partial^{k+1} \tau|^2 + \frac{2\eta A_\eta^{[k]}}{\rho} |\partial^{k+1} w|^2 \right)$$

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma \nu \mu} e^{-\varkappa \tau} (c_{\sigma \nu \mu} \partial_\tau^\sigma \lambda + c'_{\sigma \nu \mu} \partial_\tau^\sigma \eta) \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma \nu \mu \iota \mathcal{R}} c_{\sigma \nu \mu \iota \mathcal{R}} \Pi_\nu^{(k+1)} \Pi_{\mu \iota \mathcal{R} \sigma}^{(k+1)} + \widehat{\Sigma}_{\tilde{\gamma}}^{[k]}$$

where $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1 \leq |\alpha| \leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1 \leq |\alpha| \leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, with

$$\Pi_\nu^{(k+1)} = e^{(1-a_k+\varkappa)\tau/2} \prod_{1 \leq |\alpha| \leq k+1} (\partial^\alpha \tau)^{\nu_\alpha} \prod_{0 \leq |\alpha| \leq k+1} (\partial^\alpha w)^{\nu'_\alpha}$$

where w denotes any of its components w_1, \dots, w_n and μ and ν must be such that

$$\sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\nu_\alpha + \nu'_\alpha) = \sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\mu_\alpha + \mu'_\alpha) = k+1, \quad \sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leq 1$$

Higher order entropy estimates (8)

- **Governing equation for $\tilde{\gamma}^{[k]}$ (continued)**

The non strictly differential terms $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$ are given by

with
$$\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)} = \tilde{\Pi}_{\mu}^{(k+1,l)} \mathcal{R} \left(e^{-\varkappa\tau} \partial_{\tau}^{\sigma} \eta \tilde{\Pi}_{\iota}^{(k+1,k+1-l)} \right)$$

$$\tilde{\Pi}_{\mu}^{(k+1,l)} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}} \prod_{1 \leq |\alpha| \leq k+1} (\partial^{\alpha} \tau)^{\mu_{\alpha}} \prod_{0 \leq |\alpha| \leq k+1} (\partial^{\alpha} w)^{\mu'_{\alpha}}$$

$$\tilde{\Pi}_{\iota}^{(k+1,k+1-l)} = e^{(1-a_k+\varkappa)\frac{(k+1-l)\tau}{2(k+1)}} \prod_{1 \leq |\alpha| \leq k+1} (\partial^{\alpha} \tau)^{\iota_{\alpha}} \prod_{0 \leq |\alpha| \leq k+1} (\partial^{\alpha} w)^{\iota'_{\alpha}}$$

where $\nu = (\nu_{\alpha}, \nu'_{\alpha})_{1 \leq |\alpha| \leq k+1}$, $\mu = (\mu_{\alpha}, \mu'_{\alpha})_{1 \leq |\alpha| \leq k}$, $\iota = (\iota_{\alpha}, \iota'_{\alpha})_{1 \leq |\alpha| \leq k}$, $0 \leq \sigma \leq k$, and

$$\sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\mu_{\alpha} + \mu'_{\alpha}) = l, \quad \sum_{1 \leq |\alpha| \leq k+1} |\alpha| (\iota_{\alpha} + \iota'_{\alpha}) = k + 1 - l, \quad \sum_{|\alpha|=k+1} (\nu_{\alpha} + \nu'_{\alpha} + \mu_{\alpha} + \mu'_{\alpha} + \iota_{\alpha} + \iota'_{\alpha}) \leq 1$$

for $0 \leq l \leq k$, $\mathcal{R} = e^{-\theta\tau} R_i R_j e^{\theta\tau}$, $1 \leq i, j \leq n$, with $2\theta = a_k + \varkappa + l(1 - a_k + \varkappa)/(k + 1)$ and we have

$$\hat{\Sigma}_{\tilde{\gamma}}^{[k]} = \frac{A\eta^{[k]}}{\rho} \left(\frac{\lambda}{c_v} - \eta \right) e^{(1-a_k)\tau} \sum_{|\alpha|=k} \frac{k!}{\alpha!} w \cdot \partial^{\alpha} w \partial^{\alpha} \tau$$

Higher order entropy estimates (9)

- **Governing equation for $\tilde{\gamma}^{[k]}$ (continued)**

The term $\omega_{\tilde{\gamma}}^{[k]}$ is given by

$$\omega_{\tilde{\gamma}}^{[k]} e^{-(1-2\kappa+a_{k-1}-a_k)\tau/2} = \sum_{\nu\mu} c_{\nu\mu} \Pi_{\nu}^{(k)} \Pi_{\mu}^{(k+1)} + \sum_{\nu\mu\iota\mathcal{R}} c_{\nu\mu\iota\mathcal{R}} \Pi_{\nu}^{(k)} \Pi_{\mu\iota\mathcal{R}}^{(k+1)}$$

where $\sum_{1 \leq |\alpha| \leq k} |\alpha|(\nu_{\alpha} + \nu'_{\alpha}) = k$, $\sum_{1 \leq |\alpha| \leq k} |\alpha|(\mu_{\alpha} + \mu'_{\alpha}) = k + 1$, and $\sum_{|\alpha|=k+1} (\mu_{\alpha} + \mu'_{\alpha}) = 0$. For the non strictly differential terms

$$\Pi_{\mu\iota\mathcal{R}}^{(k+1)} = \tilde{\Pi}_{\mu}^{(k+1,l)} \mathcal{R}(\tilde{\Pi}_{\iota}^{(k+1,k+1-l)})$$

we have

$$\sum_{1 \leq |\alpha| \leq k} |\alpha|(\nu_{\alpha} + \nu'_{\alpha}) = k, \quad \sum_{1 \leq |\alpha| \leq k} |\alpha|(\mu_{\alpha} + \mu'_{\alpha}) = l, \quad \sum_{1 \leq |\alpha| \leq k} |\alpha|(\iota_{\alpha} + \iota'_{\alpha}) = k + 1 - l$$

and $\sum_{|\alpha|=k+1} (\mu_{\alpha} + \mu'_{\alpha} + \iota_{\alpha} + \iota'_{\alpha}) = 0$ for some $0 \leq l \leq k$ and the singular operator \mathcal{R} is in the form $T^{-\theta} R_i R_j T^{\theta}$ with $1 \leq i, j \leq n$ and $2(k+1)\theta = 2(k+1) - (k+1-l)(1-a_k+\kappa)$.

Higher order entropy estimates (10)

- **Quantity χ**

$$\chi = \|\log T\|_{BMO} + \left\| \frac{v}{\sqrt{T}} \right\|_{L^\infty} = \|\tau\|_{BMO} + \|w\|_{L^\infty}$$

- **Conditional entropicity for $\gamma^{[k]}$**

Let $k \geq 1$ and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}| dx \leq c \chi \int_{\mathbb{R}^n} \pi_\gamma^{[k]} dx$$

- **Sketch of the Proof**

The quantities $T^{\sigma-\varkappa} \partial_T^\sigma \lambda$ and $T^{\sigma-\varkappa} \partial_T^\sigma \eta$ are uniformly bounded and for χ small enough the operators $T^\theta R_i R_j T^{-\theta}$ are continuous over L^2 and

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \leq c \chi^{N_\nu-1} \left(\|T^{\frac{\theta}{2}} \frac{\partial^{k+1} T}{T}\|_{L^2} + \|T^{\frac{\theta}{2}} \frac{\partial^{k+1} v}{\sqrt{T}}\|_{L^2} \right)$$

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \|\Pi_\mu^{(k+1)}\|_{L^2} \leq c \chi^{N_\nu+N_\mu-2} \int_{\mathbb{R}^n} \pi_\gamma^{[k]} dx$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k, n)$, and $N_\nu + N_\mu - 2 = \sum_{1 \leq |\alpha| \leq k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) - 2 \geq 1$

Higher order entropy estimates (11)

- **Conditional entropicity for $\tilde{\gamma}^{[k]}$**

Let $k \geq 1$ and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}| dx \leq c \chi \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} dx$$

- **Sketch of the Proof**

Similar proof than for $\gamma^{[k]}$ and use of weighted interpolation for the estimates

$$\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\|_{L^2} \leq c\chi^{N_\mu+N_\iota-1} \left(\|e^{\frac{\theta\sigma}{2}} \partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\sigma}{2}} \partial^{k+1}w\|_{L^2} \right)$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k, n)$ so that finally

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\|_{L^2} \leq c\chi^{N_\nu+N_\mu+N_\iota-2} \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} dx$$

where $N_\nu + N_\mu + N_\iota - 2 = \sum_{1 \leq |\alpha| \leq k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) - 2 \geq 1$

Higher order entropy estimates (12)

- **Corollary**

Let $k \geq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]} dx + (1 - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} dx \leq \int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}| dx$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[k]} dx + (1 - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} dx \leq \int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}| dx$$

- **Remark**

Letting $\rho \underline{b}_k = \underline{a} \min(A_{\lambda}^{[k]}/A_{\lambda}^{[k+1]}, A_{\eta}^{[k]}/A_{\eta}^{[k+1]})$ and $\rho \bar{b}_k = \bar{a} \max(A_{\lambda}^{[k]}/A_{\lambda}^{[k+1]}, A_{\eta}^{[k]}/A_{\eta}^{[k+1]})$ we have

$$2\underline{b}_k \gamma^{[k+1]} \leq \pi_{\tilde{\gamma}}^{[k]} T^{-(a_{k+1} - a_k + \varkappa)} \leq 2\bar{b}_k \gamma^{[k+1]}$$

$$2\underline{b}_k \tilde{\gamma}^{[k+1]} \leq \pi_{\tilde{\gamma}}^{[k]} e^{-(a_{k+1} - a_k + \varkappa)\tau} \leq 2\bar{b}_k \tilde{\gamma}^{[k+1]}$$

- **Parallel with heat equation**

Letting $\zeta^{[k]} = |\partial^k u|^2$ we have $\partial_t \int_{\mathbb{R}^n} \zeta^{[k]} dx + 2 \int_{\mathbb{R}^n} |\partial_x^2 u|^{k+1} dx = 0$

Higher order entropy estimates (13)

- **Estimates of convective terms**

Let $k \geq 1$ and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\omega_{\gamma}^{[k]}| dx \leq c\chi \sup_{\mathbb{R}^n} \{T^{(1-2\kappa+a_{k-1}-a_k)/2}\} \left(\int_{\mathbb{R}^n} \pi_{\gamma}^{[k-1]} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \pi_{\gamma}^{[k]} dx \right)^{\frac{1}{2}}$$

$$\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}| dx \leq c\chi \sup_{\mathbb{R}^n} \{e^{(1-2\kappa+a_{k-1}-a_k)\tau/2}\} \left(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k-1]} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} dx \right)^{\frac{1}{2}}$$

- **Estimates for $\gamma^{[0]}$ and $\tilde{\gamma}^{[0]}$**

Let $0 < a_0 \leq 1$ then $\gamma^{[0]} = \tilde{\gamma}^{[0]} \geq 0$ and there exists constants $\delta_0 > 0$, \underline{b}'_0 and c such that for $\chi < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]} dx + \underline{b}'_0 \int_{\mathbb{R}^n} \pi_{\gamma}^{[0]} dx \leq 0$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]} dx + (\underline{b}'_0 - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]} dx \leq 0$$

Higher order entropy estimates (14)

- **Natural scale of temperature weights**

Simplified coefficients $A_\lambda^{[k]} = 1$, and $A_\eta^{[k]} = 1/r$, $k \geq 0$

Eliminate the $\sup T^z$ factors from convective terms estimates

$$a_k = a_0 + k(1 - 2\kappa), \quad k \geq 0$$

Corresponds to the scale given by the Kinetic Theory since $(\eta/\rho\sqrt{rT})^{(2k)} \sim 1/T^{k(1-2\kappa)}$

There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]} dx + (2\underline{b} - c\chi) \int_{\mathbb{R}^n} T^{1-\kappa} \gamma^{[k+1]} dx \leq c\chi \int_{\mathbb{R}^n} T^{1-\kappa} \gamma^{[k]} dx$$

- **Higher order entropy estimates**

Let (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\underline{b} = \min(\underline{b}_0, \underline{a}/\rho)$ and $\delta_N(k, n)$ such that for $\chi < \delta_N$ we have

$$\partial_t \int_{\mathbb{R}^n} (\gamma^{[0]} + \gamma^{[1]} + \dots + \gamma^{[k]}) dx + \underline{b} \int_{\mathbb{R}^n} T^{1-\kappa} (\gamma^{[1]} + \gamma^{[2]} + \dots + \gamma^{[k+1]}) dx \leq 0$$

Higher order entropy estimates (15)

- **Uniform scale of temperature weights**

Simplified coefficients $A_\lambda^{[k]} = 1$, and $A_\eta^{[k]} = 1/r$, $k \geq 0$ and simplified temperature exponents

$$a_k = a_0, \quad k \geq 0$$

Control of the $\sup T^z$ factors from convective terms estimates with $z < 0$ and $T \geq T_{\min} > 0$ and this requires $1 - 2\chi \leq 0$. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]} dx + (2\underline{b} - c\chi) \int_{\mathbb{R}^n} T^\chi \gamma^{[k+1]} dx \leq c\chi \int_{\mathbb{R}^n} T^\chi \gamma^{[k]} dx$$

- **Higher order entropy estimates**

Let (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\underline{b} = \min(\underline{b}_0, \underline{a}/\rho)$ and $\delta_U(k, n)$ such that for $\chi < \delta_N$ we have

$$\partial_t \int_{\mathbb{R}^n} (\gamma^{[0]} + \gamma^{[1]} + \dots + \gamma^{[k]}) dx + \underline{b} \int_{\mathbb{R}^n} T^\chi (\gamma^{[1]} + \gamma^{[2]} + \dots + \gamma^{[k+1]}) dx \leq 0$$

$$\partial_t \int_{\mathbb{R}^n} (\tilde{\gamma}^{[0]} + \tilde{\gamma}^{[1]} + \dots + \tilde{\gamma}^{[k]}) dx + \underline{b} \int_{\mathbb{R}^n} T^\chi (\tilde{\gamma}^{[1]} + \tilde{\gamma}^{[2]} + \dots + \tilde{\gamma}^{[k+1]}) dx \leq 0$$

Higher order entropy estimates (16)

- **Condition for entropicity**

The quantity

$$\chi = \|\log T\|_{BMO} + \left\| \frac{v}{\sqrt{T}} \right\|_{L^\infty} = \|\tau\|_{BMO} + \|w\|_{L^\infty}$$

is invariant under various change of scales

The variables τ and w can be seen as renormalized variables and also naturally arise in Maxwellian distribution functions

We have formally $\chi = \mathcal{O}(\text{Ma})$ for small Mach number flows

Application to asymptotic stability (1)

- **Incompressible model with temperature dependent transport coefficients**
- **Further assumptions**

Uniform scale of temperature weights with uniform logarithmic scaling

$$a_k = 1, \quad k \geq 0$$

The transport coefficients satisfy the natural assumptions given by the Kinetic Theory with

$$\frac{1}{2} \leq \varkappa$$

Flow spanning the whole space and ‘Constant at infinity’

We denote by v the combined unknown $v = (v, T)$ and $v_\infty = (0, T_\infty)$ the equilibrium point

We denote by $\mathcal{O}_v = \mathbb{R}^n \times (0, \infty)$ the natural domain for the variable v , where $n \geq 2$.

Application to asymptotic stability (2)

- **Local existence**

Let $n \geq 2$ and $l \geq [n/2] + 2$ be integers and let $b > 0$ be given. Let \mathcal{O}_0 be an open bounded convex set such that $\overline{\mathcal{O}_0} \subset \mathcal{O}_v$, d_1 with $0 < d_1 < d(\overline{\mathcal{O}_0}, \partial\mathcal{O}_v)$, and define $\mathcal{O}_1 = \{v \in \mathcal{O}_v; d(v, \overline{\mathcal{O}_0}) < d_1\}$.

There exists $\bar{t} > 0$ small enough, which only depend on \mathcal{O}_0 , d_1 , and b , such that for any v_0 with $\|v_0 - v_\infty\|_{H^l} < b$ and $v_0 \in \overline{\mathcal{O}_0}$, there exists a unique local solution $v = (v, T)$ to the equation system with initial condition

$$v(0, x) = v_0(x)$$

such that

$$v(t, x) \in \mathcal{O}_1$$

and

$$v - v_\infty \in C^0([0, \bar{t}], H^l(\mathbb{R}^n)) \cap C^1([0, \bar{t}], H^{l-2}(\mathbb{R}^n)) \cap L^2((0, \bar{t}), H^{l+1}(\mathbb{R}^n))$$

In addition, denoting for short $v(t) = v(t, \cdot)$, there exists $C(\mathcal{O}_0, d_1, b) > 0$ such that

$$\sup_{0 \leq s \leq \bar{t}} \|v(s) - v_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|v(s) - v_\infty\|_{H^{l+1}}^2 ds \leq C \|v_0 - v_\infty\|_{H^l}^2$$

Application to asymptotic stability (3)

- **Sketch of the Proof**

Solutions are fixed points $\tilde{\mathbf{v}} = \mathbf{v}$ of the linear system of equations in $\tilde{\mathbf{v}} = (\tilde{v}, \tilde{T})$

$$\partial_t(\rho\tilde{v}) - \mathbb{P}(\partial_x \cdot (\eta(T) \partial_x \tilde{v})) = \mathbb{P}(f_v(\mathbf{v}, \partial_x \mathbf{v}))$$

$$\partial_t(\rho c_v \tilde{T}) - \partial_x \cdot (\lambda(T) \partial_x \tilde{T}) = f_T(\mathbf{v}, \partial_x \mathbf{v})$$

with $f_v = -\partial_x \cdot (\rho v \otimes v) + \partial_T \eta(T) \partial_x T \cdot \partial_x v^t$ and $f_T = -\partial_x \cdot (\rho c_v T v) + \frac{1}{2} \eta(T) d : d$. We have the estimates

$$\|\tilde{\mathbf{v}}(t) - \mathbf{v}_\infty\|_{H^k}^2 + \int_0^t \|\tilde{\mathbf{v}}(s) - \mathbf{v}_\infty\|_{H^{k+1}}^2 ds \leq C_1^2 \exp(C_2(t + M_1\sqrt{t})) (\|\mathbf{v}_0 - \mathbf{v}_\infty\|_{H^k}^2 + C_2 \int_0^t \|f(s)\|_{H^{k-1}}^2 ds)$$

where $C_1 = C_1(\mathcal{O}_1)$, $C_2 = C_2(\mathcal{O}_1, M)$ and

$$\sup_{0 \leq s \leq \bar{t}} \|\mathbf{v}(s) - \mathbf{v}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathbf{v}(s) - \mathbf{v}_\infty\|_{H^{l+1}}^2 ds \leq M^2 \quad \int_0^{\bar{t}} \|\partial_t \mathbf{v}(s)\|_{H^{l-1}}^2 ds \leq M_1^2$$

These a priori estimates for linear equations are obtained by deriving the governing equations, multiplying by the derivative of the solution, and using the properties of the Leray projector \mathbb{P} . Existence of such solutions $\tilde{\mathbf{v}}$ to the linear equations are obtained from a priori estimates by Galerkin approximations.

Application to asymptotic stability (4)

- **Regularity of the local solution**

Assume that for some $k \geq l$ we have $v_0 - v_\infty \in H^k$. Then

$$v - v_\infty \in C^0([0, \bar{t}], H^k(\mathbb{R}^n)) \cap C^1([0, \bar{t}], H^{k-2}(\mathbb{R}^n)) \cap L^2((0, \bar{t}), H^{k+1}(\mathbb{R}^n))$$

In particular, v is smooth when $v_0 - v_\infty \in H^k(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

- **Diffeomorphism \mathcal{F}**

Denote by $\mathcal{F} : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}^{n+1}$ the application defined by $\mathcal{F}(v) = w = (w, \tau)$, that is,

$\mathcal{F}(v, T) = (w, \tau) = (v/\sqrt{T}, \log T)$. Then \mathcal{F} is a C^∞ diffeomorphism and its jacobian matrix reads

$$\partial_v \mathcal{F} = \begin{pmatrix} \frac{\mathbb{I}}{\sqrt{T}} & -\frac{1}{2} \frac{v}{T^{3/2}} \\ 0 & \frac{1}{T} \end{pmatrix}$$

In addition, for any $M_w > 0$, $M_\tau > 0$, defining $\tilde{\mathcal{O}} = (-M_w, M_w)^n \times (-M_\tau, M_\tau)$, the corresponding open set $\mathcal{O} = \mathcal{F}^{-1}(\tilde{\mathcal{O}})$ is convex.

Application to asymptotic stability (5)

- **Transport of the local solution**

Let $M_w > 0$, $M_\tau > 0$, $\tilde{\mathcal{O}}_0 = (-M_w, M_w)^n \times (-M_\tau, M_\tau)$, $\mathcal{O}_0 = \mathcal{F}^{-1}(\tilde{\mathcal{O}}_0)$, let $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_v)$, denote $\mathcal{O}_1 = \{v \in \mathcal{O}_v; d(v, \overline{\mathcal{O}}_0) < d_1\}$, and select an arbitrary $b > 0$. The local solution built with \mathcal{O}_0 , d_1 , and b is such that

$$w - w_\infty \in C^0([0, \bar{t}], H^l(\mathbb{R}^n)) \cap C^1([0, \bar{t}], H^{l-2}(\mathbb{R}^n)) \cap L^2((0, \bar{t}), H^{l+1}(\mathbb{R}^n))$$

and there exists $C > 0$ which only depend on \mathcal{O}_0 , d_1 , and b , such that

$$\sup_{0 \leq s \leq \bar{t}} \|w(s) - w_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|w(s) - w_\infty\|_{H^{l+1}}^2 ds \leq C \|w_0 - w_\infty\|_{H^l}^2$$

Moreover, the kinetic entropy estimators are such that $\Gamma^{[l]}, \tilde{\Gamma}^{[l]} \in C([0, \bar{t}], L^1(\mathbb{R}^n))$.

- **Estimates for w**

There exists a constant $\underline{c}_\Gamma(T_{\min})$ such that for any $k \geq 0$ and any w with $w - w_\infty \in H^k$ we have

$$\underline{c}_\Gamma \|w - w_\infty\|_{H^k}^2 \leq \int_{\mathbb{R}^n} \tilde{\Gamma}^{[k]} dx$$

Application to asymptotic stability (6)

• Global solutions

Let $n \geq 2$ and $l \geq [n/2] + 2$ be integers. Assume that the coefficients λ and η are such that $\varkappa \geq 1/2$. There exists $\delta_\Gamma(l, n, T_{\min}) > 0$ such that for T_0 and v_0 satisfying $T_{\min} \leq \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $v_0 - v_\infty \in H^k$, $k \in \mathbb{N}$, and

$$\int_{\mathbb{R}^n} \tilde{\Gamma}_0^{[l]} dx \leq \delta_\Gamma$$

there exists a unique global solution $v = (v, T)$ such that

$$\begin{aligned} v - v_\infty, w - w_\infty &\in C([0, \infty), H^l(\mathbb{R}^n)) \cap C^1([0, \infty), H^{l-2}(\mathbb{R}^n)) \\ \partial_x v, \partial_x w &\in L^2((0, \infty), H^l(\mathbb{R}^n)) \end{aligned}$$

and we have the estimates

$$\int_{\mathbb{R}^n} \tilde{\Gamma}^{[l]} dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} T^\varkappa (\tilde{\Gamma}^{[l+1]} - \tilde{\gamma}^{[0]}) dx dt \leq \int_{\mathbb{R}^n} \tilde{\Gamma}_0^{[l]} dx$$

where $\underline{b} = \min(\underline{b}_0, \underline{a}/\rho)$. Furthermore, this solution is smooth and we have

$$\lim_{t \rightarrow \infty} \|v(t, \cdot) - v_\infty\|_{L^\infty} = 0$$

Application to asymptotic stability (7)

- **Sketch of the Proof**

We have the inequalities

$$\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} \leq \|\tau - \tau_\infty\|_{L^\infty} + \|w\|_{L^\infty} \leq c_0 \|W - W_\infty\|_{H^1_0} \leq \frac{c_0}{\sqrt{c_\Gamma}} \int_{\mathbb{R}^n} \tilde{\Gamma}^{[l]} dx$$

We set

$$\delta_\Gamma = \frac{\delta_U^2}{4c_0^2} c_\Gamma$$

so that

$$\int_{\mathbb{R}^n} \tilde{\Gamma}^{[l]} dx < \delta_\Gamma \implies \chi < \delta_U/2$$

Higher order entropy estimates yields that

$$\int_{\mathbb{R}^n} \tilde{\Gamma}^{[l]} dx \leq \int_{\mathbb{R}^n} \tilde{\Gamma}_0^{[l]} dx$$

Local existence used repeatedly

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Conclusion and future work (1)

- **Incompressible models**

 - Asymptotic analysis for small Mach numbers

- **Compressible models**

 - Higher order entropy estimates

 - Application to asymptotic stability

 - Asymptotic analysis for small Mach numbers

- **Zero Mach number models**

 - Higher order entropy estimates

 - Application to asymptotic stability

- **Multicomponent flows**