Representation formulas for the field perturbations caused by low volume fraction inhomogeneities. Applications to electromagnetic imaging.

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In this talk I shall discuss some recent work concerning representation formulas for perturbations in the electromagnetic fields caused by low volume fraction inhomogeneities, and the practical use of these formulas for the purpose of identifying and "reconstructing" the inhomogeneities from relatively few boundary field measurements. In the simplest case we consider a conducting object that occupies a bounded, smooth domain $\Omega \subset I\!R^m$. $\gamma_0(\cdot)$ denotes the smooth background conductivity, that is, the conductivity in the absence of any inhomogeneities. We suppose that

$$0 < c_0 \leq \gamma_0(x) \leq C_0 < \infty, \quad x \in \Omega$$

for some fixed constants c_0 and C_0. The function ψ denotes the imposed boundary current. It suffices that $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi \; ds \; = 0$. The background voltage potential, U, is the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma_0(x) \nabla U) = 0 \quad \text{in } \Omega \ , \\ \gamma_0(x) \frac{\partial U}{\partial n} = \psi \ \text{ on } \partial\Omega \quad . \end{aligned}$$

Here n denotes the unit outward normal to the domain Ω.
Let ω_ϵ denote a set of inhomogeneities inside Ω. The geometric assumptions about the set of inhomogeneities are very simple: we suppose the set ω_ϵ is measurable, and separated away from the boundary, (*i.e.*, $\text{dist}(\omega_\epsilon, \partial\Omega) > d_0 > 0$). Most importantly, we suppose that $0 < |\omega_\epsilon|$ gets arbitrarily small, where $|\omega_\epsilon|$ denotes the Lebesgue measure of ω_ϵ. Let $\hat{\gamma}_\epsilon$ denote the conductivity profile in the presence of the inhomogeneities. The function $\hat{\gamma}_\epsilon$ is equal to γ_0, except on the set of inhomogeneities; on the set of inhomogeneities we suppose that $\hat{\gamma}_\epsilon$ equals the restriction of some other smooth function, $\gamma_1 \in C^\infty(\overline{\Omega})$, with

$$0 < c_1 \leq \gamma_1(x) \leq C_1 < \infty, \quad x \in \Omega \quad .$$

In other words

$$\hat{\gamma}_\epsilon(x) = \begin{cases} \gamma_0(x), & x \in \Omega \setminus \omega_\epsilon \\ \gamma_1(x), & x \in \omega_\epsilon \end{cases}$$

The voltage potential in the presence of the inhomogeneities is denoted $u_\epsilon(x)$. It is the solution to

$$\begin{aligned} \nabla \cdot (\hat{\gamma}_\epsilon(x) \nabla u_\epsilon) = 0 \quad \text{in } \Omega \ , \\ \hat{\gamma}_\epsilon(x) \frac{\partial u_\epsilon}{\partial n} = \psi \ \text{ on } \partial\Omega \quad . \end{aligned}$$

We normalize both U and u_ϵ by requiring that

$$\int_{\partial\Omega} U ds = 0 \ , \quad \text{and} \quad \int_{\partial\Omega} u_\epsilon ds = 0 \quad .$$

We note that the individual voltages U and u_ϵ need not be smooth (or even continuous) on $\partial\Omega$, however, the difference $u_\epsilon - U$ is smooth in a neighborhood of $\partial\Omega$, due to the regularity of γ_0, and the fact that ω_ϵ is strictly interior. The simplest version of our perturbation representation formulas asserts that (after the possible extraction of a subsequence)

$$\begin{aligned} (u_{\epsilon_n} - U)(y) \quad &= \quad |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial U}{\partial x_i} \frac{\partial N}{\partial x_j}(x,y) \; d\mu(x) \\ &\quad + o(|\omega_{\epsilon_n}|) \qquad y \in \partial\Omega \quad . \end{aligned} \tag{1}$$

Here $N(x,y)$ is a fundamental solution (a Neumann function) corresponding to the smooth background conductivity. M is a (symmetric, positive definite) matrix valued function, and μ is a probability measure

[6],[8]. It is possible to establish optimal bounds for the set of possible matrices M, [6], [7]. To be precise

$$\begin{aligned} & \min\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\}\ |\xi|^2 \le M_{ij}(x)\xi_i\xi_j \le \max\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\}\ |\xi|^2 \quad , \\ & \text{Trace } M(x) \le m-1+\frac{\gamma_0(x)}{\gamma_1(x)} \quad , \qquad \text{and} \\ & \text{Trace } M^{-1}(x) \le m-1+\frac{\gamma_1(x)}{\gamma_0(x)} \quad , \end{aligned} \tag{2}$$

μ almost everywhere in the set $\{\ x\ :\ \gamma_0(x) \neq \gamma_1(x)\ \}$. Quite strikingly (but on second thought, maybe not so surprisingly) these bounds are the same as one may obtain for the (appropriately rescaled) first variation with respect to volume fraction of the standard effective tensor that results from the mixture of two material components [3], [11]. In the case when ω_ϵ is of the form $\omega_\epsilon = \cup_{j=1}^N (z_j + \epsilon B_j)$ (a finite collection of diametrically small inhomogeneities) then the probability measure μ becomes a sum of delta masses

$$\mu = \sum_{j=1}^N \alpha_j \delta_{z_j} = \sum_{j=1}^N \frac{|B_j|}{\sum |B_j|}\delta_{z_j} \quad ,$$

and (1) takes the form

$$\begin{aligned} (u_\epsilon - U)(y) \quad &= \quad |\omega_\epsilon| \sum_{j=1}^N \alpha_j (\gamma_1 - \gamma_0)(z_j) M^{(j)} \nabla U(z_j) \cdot \nabla_x N(z_j, y) \\ &\qquad + o(|\omega_\epsilon|) \qquad y \in \partial\Omega \end{aligned} \tag{3}$$

(no extraction of a subsequence is necessary). In the case when ω_ϵ is a finite collection of uniformly thin sheets with "mid-surfaces", σ_j, $1 \le j \le N$, and thickness ϵ, then the probability measure becomes a sum of surface measures

$$\mu = \sum_{j=1}^N \frac{1}{\sum |\sigma_j|} ds|_{\sigma_j} \quad ,$$

and (1) reads

$$\begin{aligned} (u_\epsilon - U)(y) \quad &= \quad \epsilon \sum_{j=1}^N \int_{\sigma_j} (\gamma_1 - \gamma_0)(x) M(x) \nabla U(x) \cdot \nabla_x N(x, y)\ ds_x \\ &\qquad + o(\epsilon) \qquad y \in \partial\Omega \quad . \end{aligned} \tag{4}$$

In this case the polarization tensor $M(x)$ has eigenvalues 1 in the directions tangent to the mid-surfaces, and eigenvalue $\gamma_0(x)/\gamma_1(x)$ in the orthogonal direction [4]. This tensor is "extreme" in the sense of the inequalities (2), and it is possible to give a fairly simple proof of the representation formula (4), using exactly that fact [7].

In the talk I shall briefly describe some of the techniques used to derive representation formulas such as (1)–(4) (and higher order generalizations [1]). I shall also outline how specific formulas like this for $u_{\epsilon_n} - U$ allow for a quite accurate estimation of the total volume of the inhomogeneities in terms of one, two or three boundary measurements [6]. Furthermore I shall describe how methods of a "linear sampling type" in certain cases may be used to determine the probabbily measure μ from more detailed knowledge of the full Dirichlet-to-Neumann data map [5].

Representation formulas like (1) hold in much more generality than described above. I shall in particular discuss the case of extreme (infinite or zero) conductivity, and the necessary modifications [9]. I shall also describe some very recent results concerning the Helmholtz Equation (and more generally, the time-harmonic Maxwell's Equations) with particular emphasis on the change of the formulas with respect to frequency [2], [10]. These latter results are for the moment restricted to the case when $\omega_\epsilon = \cup_{j=1}^N (z_j + \epsilon B_j)$. For small and moderate frequencies (of the order smaller that $1/\epsilon$) we can rigorously derive a perturbation formula very much like the zero frequency formula (3). For large frequencies the situation becomes significantly more complex, and we have so far only limited rigorous results. We have heuristically derived quite precise, approximate representation formulas, based on a combination of an appropriate Green's formula, an ansatz of the nature of geometric optics, and stationary phase arguments [10].

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