Dissipative hyperbolic systems: asymptotic behavior and numerical approximation

Roberto NATALINI, IAC-CNR

In this talk we shall consider the Cauchy problem for a general hyperbolic symmetrizable m-dimensional system of balance laws

$$u_t + \sum_{\alpha=1}^{m} (f_\alpha(u))_{x_\alpha} = g(u), \tag{1}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{2}$$

where $u = (u_1, u_2) \in \Omega \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with $n_1 + n_2 = n$. We also assume that there are n_1 conservation laws in the system, namely that we can take

$$g(u) = \begin{pmatrix} 0 \\ q(u) \end{pmatrix}, \quad \text{with } q(u) \in \mathbb{R}^{n_2}. \tag{3}$$

According to the general theory of hyperbolic systems of balance laws, if the flux functions f_α and the source term g are smooth enough, it is well-known that problem (1)-(2) has a unique local smooth solution, at least for some time interval $[0, T)$ with $T > 0$, if the initial data are also sufficiently smooth. In the general case, and even for nice initial data, smooth solutions may break down in finite time, due to the appearance of singularities, either discontinuities or blow-up. Despite these general considerations, sometimes dissipative mechanisms due to the source term can prevent the formation of singularities, at least for some restricted classes of initial data, as observed for many models which arise to describe physical phenomena. A typical and well-known example is given by the compressible Euler equations with damping, see [8, 6] for the 1-dimensional case and [11] for an interesting 3-dimensional extension.
Recently, in [5], it was proposed a quite general framework of sufficient conditions which guarantee the global existence in time of smooth solutions. Actually, for the systems which are endowed with a strictly convex entropy function $\mathcal{E} = \mathcal{E}(u)$, a first natural assumption is the *entropy dissipation condition*, see [4], namely for every u, $\overline{u} \in \Omega$, with $g(\overline{u}) = 0$,

$$(\mathcal{E}'(u) - \mathcal{E}'(\overline{u})) \cdot g(u) \leq 0.$$

Unfortunately, it is easy to see that this condition is too weak to prevent the formation of singularities. A quite natural supplementary condition can be imposed to entropy dissipative systems, following the classical approach by Shizuta and Kawashima [7, 10], which in the present case reads

$$Ker\ Dg(\overline{u}) \cap \{\text{eigenspaces of } \sum_{\alpha=1}^{m} Df_\alpha(\overline{u})\xi_\alpha\} = \{0\}, \tag{4}$$

for every $\xi \in \mathbb{R}^m \setminus \{0\}$ and every $\overline{u} \in \Omega$, with $g(\overline{u}) = 0$. It is possible to prove that this condition, which is satisfied in many interesting examples, is also sufficient to establish a general result of global existence for small perturbations of equilibrium constant states, see [5].
In this talk we shall present some new results, obtained in [2], about the asymptotic behavior in time of the global solutions, then always assuming the existence of a strictly convex entropy and condition (4), and some related numerical schemes, investigated in [1], which are designed to give a more accurate numerical approximation of these problems for large times. Observe that in this generality, previous results can be found only in few recent papers, as for instance in [9], where the stability of constant equilibrium states was established for general hyperbolic systems in one space dimension, but only for zero-mass perturbations.
First let us describe our approach in the one dimensional case. Our starting point is a careful and refined analysis of the behavior of the Green function for the linearized problem, which is decomposed in three main terms. The first term, the diffusive one, consists of heat kernels moving along the characteristic directions of the local relaxed hyperbolic problem; the singular part consists of exponentially decaying δ-functions along the characteristic directions of the full system. Finally the remainder term decays faster than the first one. This description is crucial in the analysis of the asymptotic behavior. Let us better explain this point. Actually, we show that solutions have canonical projections on two different

components: the **conservative** part and the **dissipative** part. The first one, which loosely speaking corresponds to the conservative part of equations in (1), decays in time like the heat kernel, since it corresponds to the diffusive part of the Green function. On the other side, the dissipative part is strongly influenced by the dissipation and decays at a rate $t^{-\frac{1}{2}}$ faster of the conservative one. To establish this result, we use the Duhamel principle and the Green kernel estimates, in the spirit of the Kawashima approach for the hyperbolic-parabolic equations [7]. Unfortunately, with respect to that result, here there is a severe obstruction given by the lack of decay of the source term, when convoluted with the Green kernel. This is not the case for convective or diffusive terms, since they are derivative terms, so having a better decay, of an order $\frac{1}{2}$ for every derivative. However, in the present work we have shown that there is a structural algebraic **compatibility** between the Green kernel and the conservative structure of the system, by decomposing the kernel according to different linear projectors, which yields the **cancellation** of its highest order and slowly decaying interactions with the source term.
In the multidimensional case, the explicit form of the Green function cannot in general be expressed, and we have to relay directly on the Fourier coordinates. Thus the separation of the Green kernel into various part is done at the level of solution operator $\Gamma(t)$ acting on $L^2(R^m, R^n)$, or $L^1 \cap L^2(R^m, R^n)$. This allow to perform L^p linear decay estimates, for $p \geq 2$.
Concerning the nonlinear case, we first prove the decay results for both the conservative part u_c and the dissipative part u_d of the solution. For $s > 0$, let $E_s = \max\Big\{\|u(0)\|_{L^1}, \|u(0)\|_{H^s}\Big\}$.

Theorem 1 *Let $u(t) = (u_c(t), u_d(t))$ be a smooth global solution to problem* (1), (2). *Assume $E_{[m/2]+2}$ small enough and let $p \in [\min\{2, m\}, +\infty]$. The following decay estimate holds*

$$\|D^\beta u(t)\|_{L^p} \leq C \min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2}\Big\} E_{|\beta|+[m/2]+1}, \tag{5}$$

with $C = C(E_{|\beta|+\sigma})$, for σ large enough. For the dissipative part, we have a more precise estimate:

$$\|D^\beta u_d(t)\|_{L^p} \leq C \min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-1/2-|\beta|/2}\Big\} E_{|\beta|+[m/2]+1}, \tag{6}$$

Besides, we show also that the conservative variable approaches the conservative part of the solution of the corresponding linearized problem, faster that the decay of the heat kernel for $m \geq 2$. In the same spirit, another interesting result concerns the convergence to the Chapman-Enskog expansion of problem (1), which for $m \geq 2$ reads

$$w_t + \sum_{\alpha=1}^{m} A_{\alpha,11}(0) w_{x_\alpha} + \sum_{\beta=1}^{m}\sum_{\alpha=1}^{m} A_{\alpha,12}(0)(D_{u_d} q(0))^{-1} A_{\beta,21}(0) w_{x_\alpha x_\beta} = 0. \tag{7}$$

In this case it is possible to prove the following result.

Theorem 2 *Let u_p be the solution of problem* (7), *with a suitable initial condition. Under the assumptions of Theorem 1, for $m \geq 2$ and $p \in [2, \infty]$, we have the following more accurate decay estimate*

$$\|D^\beta(u_c(t) - u_p(t))\|_{L^p} \leq C \min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2-1/2}\Big\} E_{|\beta|+[m/2]+1}, \tag{8}$$

with $C = C(E_{|\beta|+\sigma})$, for σ large enough.

For $m \geq 2$ the Chapman-Enskog operator is linear, while, in one space dimension, the decay of the nonlinear part has a stronger influence, and so we can only show the faster convergence towards the solution of a parabolic equation with quadratic nonlinearity.
Many physical models enter the hypotheses of these results, for instance many rotationally invariant systems, as for instance the isentropic Euler equations with damping or the relaxation system

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ (\rho v)_t + \operatorname{div}(\rho R) + \nabla\rho = 0, \\ (\rho R)_t + \nabla(\rho v) = \rho v \otimes v - \rho R. \end{cases} \tag{9}$$

In this case, we can explicitly identify the asymptotic limit as the the weakly parabolic system

$$\begin{cases} \rho_t + \mathrm{div} v = 0, \\ \\ v_t + \nabla \rho = \Delta v. \end{cases} \tag{10}$$

Actually it is possible to show, in a less elementary way, that another equivalent asymptotics is given by the same system with a term $\Delta\rho$ on the R.H.S. of the first equation.
In the last part of the talk, numerical approximations related to these asymptotic results will be discussed. The main idea is to modify standard upwinding schemes to keep into account the long-time behaviour of the solutions. For general 2×2 dissipative hyperbolic one dimensional systems, we shall introduce some schemes which are in reasingly accurate for large times, with respect to the asymptotic behavior of solutions. This property of accuracy is required in order to get better results for large time simulations when computing perturbations of some given stable states. Given a family of stable asymptotic states for a given evolutionary problem, in [3] we introduced the *Asymptotic High Order* schemes , i.e.: schemes which are high-order accurate with respect to the local truncation error, when restricted to every element of this family. We shall show that for 2×2 dissipative hyperbolic systems it is possible to introduce AHO schemes which are compatible with the behavior predicted by the previous analysis, and also in the diffusive regime. Numerical tests will be presented and discussed.

References

[1] D. Aregba-Driollet, M. Briani, R. Natalini, *Asymptotic high-order schemes for dissipative hyperbolic systems*, in preparation.

[2] S. Bianchini, B. Hanouzet, R. Natalini, *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, IAC Report 79 (11/2005); to appear in Comm. Pure and Appl. Math.

[3] M. Briani, R. Natalini, *Asymptotic high-order schemes for integro-differential problems arising in markets with jumps*, Commun. Math. Sci. **4** (2006), 81-96.

[4] G.-Q. Chen, C.D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure and Appl. Math. **47** (1994), 787–830.

[5] B. Hanouzet and R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Ration. Mech. Anal. **169** (2003), no. 2, 89–117.

[6] L. Hsiao and T.-P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Physics **143** (1992), 599–605.

[7] S. Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), no. 1-2, 169–194.

[8] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Département de Mathématique, Université de Paris-Sud, Orsay, 1978, Publications Mathématiques d'Orsay, No. 78-02.

[9] T. Ruggeri and D. Serre, *Stability of constant equilibrium state for dissipative balance laws system with a convex entropy*. Quart. Appl. Math. 62 (2004), no. 1, 163–179.

[10] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985), no. 2, 249–275.

[11] T. Sideris, B. Thomases, and D. Wang, *Decay and singularities of solutions of the three-dimensional Euler equations with damping*, Comm. Partial Differential Equations 28 (2003), no. 3-4, 795–816.

Roberto NATALINI – `r.natalini@iac.cnr.it`
Istituto per le Applicazioni del Calcolo "M. Picone", CNR, Viale del Policlinico 137, I-00161 Roma (Italy)