Controllability and nonlinearity

Jean-Michel CORON, Département de mathématiques, Université de Paris-Sud

A usual problem in control theory is the problem of controllability: given two states, is it possible to go from the first one to the second one by means of suitable control?
Even in finite dimension, find a necessary and sufficient condition for controllability is out of reach.
One can restrict our goal to the study of local controllability. In this case the two states are close to some given equilibrium. A major method to study the local controllability around an equilibrium is to look at the controllability of the linearized control system around this equilibrium. Indeed, using the inverse mapping theorem, the controllability of this linearized control system implies the local controllability of the nonlinear control system, in any cases in finite dimension and in many cases in infinite dimension. In infinite dimension the situation can be more complicated due to some problems of "loss of derivatives" as we shall see on examples. However, classical iterative schemes, as we shall see for hyperbolic systems [7], and the Nash-Moser method as introduced by Karine Beauchard in [1] for Schrödinger (see also [2]) can allow to handle some of these cases.
When the linearized control system around the equilibrium is not controllable, the situation is more complicated. However, for finite-dimensional systems, one knows powerful tools to handle this situation. These tools rely on iterated Lie brackets. They lead to many sufficient or necessary conditions for local controllability of a nonlinear control system. We shall recall some of these conditions.
In infinite dimension, iterated Lie brackets give some interesting results. However we shall see that these iterated Lie brackets do not work so well in many interesting cases.
We present here three methods to get in some cases controllability results for some control systems modeled by partial differential equations even if the linearized control system around the equilibrium is not controllable. These methods are

1. The return method,
2. Quasi-static deformations,
3. Power series expansion.

Let us briefly describe them.
Return method. The idea of the return method goes as follows: If one can find a trajectory of the nonlinear control system such that

- It starts and ends at the equilibrium,
- The linearized control system around this trajectory is controllable,

then, in general, the implicit function theorem implies that one can go from every state close to the equilibrium to every other state close to the equilibrium. This method has been introduced in [3, 4]. It has already been used for various partial differential equations:

1. For the Euler equation of incompressible fluids; see [4, 6], and [15] by Olivier Glass,
2. For the Navier-Stokes equation; see [5, 9] and [14] by Andrei Fursikov and Oleg Imanuvilov,
3. For the Burgers equation by Thierry Horsin in [18],
4. For the Saint-Venant equation in [7],
5. For the Vlasov-Poisson equation by Olivier Glass in [16],
6. For the Schrôdinger equation by Karine Beauchard in [1]; see also [2],
7. For the isentropic Euler equation by Olivier Glass in [17].

Quasi-static deformations. The quasi-static deformation method allows one to prove in some cases that one can move from every equilibrium to every other equilibrium if these two equilibria are connected in the set of equilibria. The idea is just to move very slowly the control (quasi-static deformation) so that at each time the state is close to the curve of equilibria connecting the two given equilibria. If some of these equilibria are instable, one also uses suitable feedback laws in order to stabilize them: without these feedback laws the quasi-static deformation method would not work. This method has been introduced in [7]. It has also been used

- For a heat equation in [10],
- For a Schrödinger equation by Karine Beauchard in [1]; see also [2].
- For a wave equation in [11],
- For Couette flows by Michael Schmidt and Emmanuel Trélat in [21].

Power series expansion. In this method one does some power series expansion in order to decide whether the nonlinearity allows to move in the directions which are not controllable for the linearized control system around the equilibrium. This method has been introduced in [8] for a Korteweg-de Vries equation and also been used in [2].

In this talk we present two applications of these methods:

1. For a Saint-Venant equation in Section 1,
2. For a Schrodinger equation in Section 2.

In the next sections we only state the results which have been obtained on these two equations. During the talk we sketch the main ideas of the proofs.

1 Saint-Venant equation

In this section, we consider a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to one-dimensional horizontal moves. We assume that the horizontal acceleration of the tank is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. These physical considerations motivate the use of the Saint-Venant equation [20] (also called shallow water equation) to describe the motion of the fluid; see e.g. [12, Sec. 4.2]. Hence the dynamics equations considered are -see the paper [13] by François Dubois, Nicolas Petit and Pierre Rouchon-

$$H_t\,(t,x) + (Hv)_x\,(t,x) = 0, \tag{1}$$

$$v_t\,(t,x) + \left(gH + \frac{v^2}{2}\right)_x (t,x) = -u\,(t)\,, \tag{2}$$

$$v(t,0) = v(t,L) = 0, \tag{3}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\,(t) = u\,(t)\,, \tag{4}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t}\,(t) = s\,(t)\,, \tag{5}$$

where (see Figure 1),

- L is the length of the 1-D tank,
- $H\,(t,x)$ is the height of the fluid at time t and at the position $x \in [0,L]$,
- $v\,(t,x)$ is the horizontal water velocity of the fluid *in a referential attached to the tank* at time t and at the position $x \in [0,L]$ (in the shallow water model, all the points on the same vertical have the same horizontal velocity),
- $u\,(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

This is a control system, denoted Σ, where

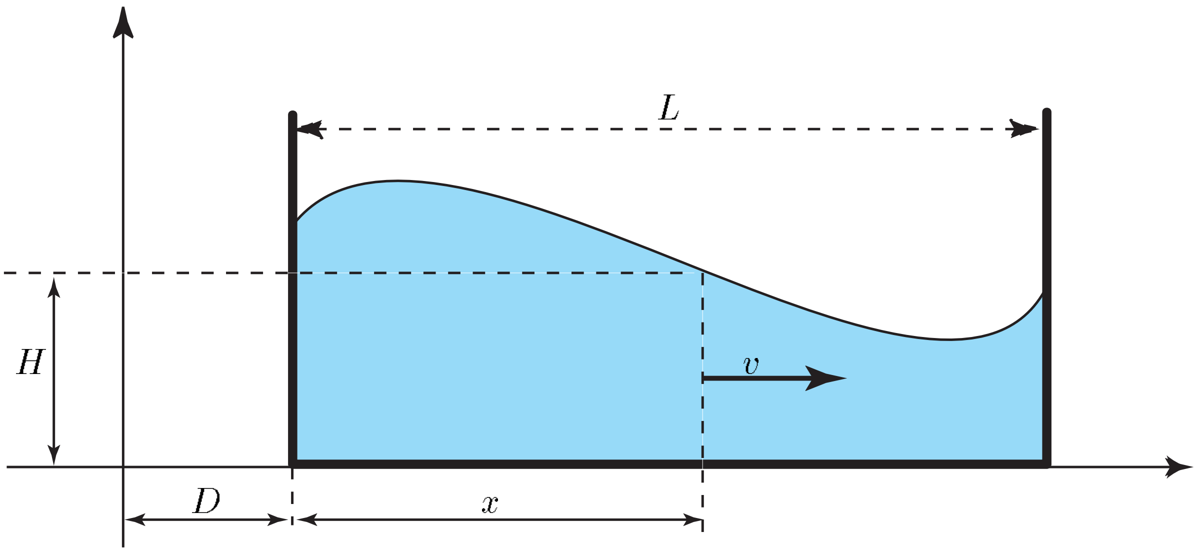

Figure 1: Fluid in the 1-D tank

- The state is $Y = (H, v, s, D)$,
- The control is $u \in \mathbb{R}$.

Our goal is to study the local controllability of this control system Σ around the equilibrium point

$$(Y_e, u_e) := ((H_e, 0, 0, 0), 0).$$

This problem has been raised by François Dubois, Nicolas Petit and Pierre Rouchon in [13].
Of course, the total mass of the fluid is conserved so that, for every solution of (1) to (3),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L H\,(t, x)\,\mathrm{d}x = 0. \tag{6}$$

(One gets (6) by integrating (1) on $[0, L]$ and by using (3) together with an integration by parts.) Moreover, if H and v are of class C^1, it follows from (2) and (3) that

$$H_x(t, 0) = H_x(t, L), \tag{7}$$

which is also $-u\,(t)\,/g$. Therefore we introduce the vector space E of functions $Y = (H, v, s, D) \in C^1([0, L]) \times C^1([0, L]) \times \mathbb{R} \times \mathbb{R}$ such that

$$H_x(0) = H_x(L), \tag{8}$$

$$v(0) = v(L) = 0, \tag{9}$$

and consider the affine subspace $\mathcal{Y} \subset E$ consisting of elements $Y = (H, v, s, D) \in E$ satisfying

$$\int_0^L H(x)\mathrm{d}x = LH_e. \tag{10}$$

The vector space E is equipped with the usual norm

$$|Y| := |H|_1 + |v|_1 + |s| + |D|,$$

where, for $w \in C^1([0, L])$,

$$|w|_1 := \mathrm{Max}\{|w(x)| + |w_x(x)|;\ x \in [0, L]\}.$$

With these notations, we can define a trajectory of the control system Σ.

Definition 1 *Let T_1 and T_2 be two real numbers satisfying $T_1 \leq T_2$. A function $(Y,u) = ((H,v,s,d),u) : [T_1,T_2] \to \mathcal{Y} \times \mathbb{R}$ is a trajectory of the control system Σ if*

(i) The functions H and v are of class C^1 on $[T_1,T_2] \times [0,L]$,

(ii) The functions s and D are of class C^1 on $[T_1,T_2]$ and the function u is continuous on $[T_1,T_2]$,

(iii) The equations (1) to (5) hold for every $(t,x) \in [T_1,T_2] \times [0,L]$.

Our main result states that the control system Σ is locally controllable around the equilibrium point (Y_e, u_e). More precisely, one has the following theorem.

Theorem 2 *([7]) There exist $T > 0$, $C_0 > 0$ and $\eta > 0$ such that, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$, and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ such that*

$$|H_0 - H_e|_1 + |v_0|_1 < \eta,\ |H_1 - H_e|_1 + |v_1|_1 < \eta,\ |s_1 - s_0| + |D_1 - s_0 T - D_0| < \eta,$$

there exists a trajectory $(Y,u) : t \in [0,T] \mapsto ((H(t), v(t), s(t), D(t)), u(t)) \in \mathcal{Y} \times \mathbb{R}$ of the control system Σ, such that

$$Y(0) = Y_0 \text{ and } Y(T) = Y_1, \tag{11}$$

and, for every $t \in [0,T]$,

$$|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| \leq C_0 \left(\sqrt{|H_0 - H_e|_1 + |v_0|_1 + |H_1 - H_e|_1 + |v_1|_1} + |s_1 - s_0| + |D_1 - s_0 T - D_0| \right). \tag{12}$$

As a corollary of this theorem, any steady state $Y_1 = (H_e, 0, 0, D_1)$ can be reached from any other steady state $Y_0 = (H_e, 0, 0, D_0)$. More precisely, one has the following corollary.

Corollary 3 *([7]) Let T, C_0 and η be as in Theorem 2. Let D_0 and D_1 be two real numbers and let $\eta_1 \in (0, \eta]$. Then there exists a trajectory*

$$\begin{array}{rcc} (Y,u) : [0, T(|D_1 - D_0| + \eta_1)/\eta_1] & \to & \mathcal{Y} \times \mathbb{R} \\ t & \mapsto & ((H(t), v(t), s(t), D(t)), u(t)) \end{array}$$

of the control system Σ, such that

$$Y(0) = (H_e, 0, 0, D_0) \text{ and } Y(T(|D_1 - D_0| + \eta_1)/\eta_1) = (H_e, 0, 0, D_1), \tag{13}$$

$$|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| \leq C_0 \eta_1,\ \forall t \in [0, T(|D_1 - D_0| + \eta_1)/\eta_1]. \tag{14}$$

2 Schrödinger equation

Let $I = (-1, 1)$. We consider the following Schrödinger control system.

$$i\psi_t(t,x) = -\psi_{xx}(t,x) - u(t)x\psi(t,x),\ (t,x) \in (0,T) \times I, \tag{15}$$

$$\psi(t,-1) = \psi(t,1) = 0,\ t \in (0,T), \tag{16}$$

$$\dot{S}(t) = u(t),\ \dot{D}(t) = S(t),\ t \in (0,T). \tag{17}$$

This is a control system, where

- The state is (ψ, S, D) with $\int_I |\psi(t,x)|^2 dx = 1$ for every $t \in [0,T]$,
- The control is the function $t \in [0,T] \mapsto u(t) \in \mathbb{R}$.

This system has been introduced by Pierre Rouchon in [19]. It models a non relativistic charged particle in a 1-D moving infinite square potential well. At time t, $\psi(t,\cdot)$ is the wave function of the particle in a frame attached to the potential well, $S(t)$ is the speed of the potential well and $D(t)$ is the displacement of the potential well. The control $u(t)$ is the acceleration of the potential well at time t. We want to control at the same time the wave function ψ, the speed S and the position D of the potential well. Let us first recall some important trajectories of the above control system when the control is 0. Let

$$\psi_n(t,x) := \varphi_n(x)e^{-i\lambda_n t}, n \in \mathbb{N}^*. \tag{18}$$

Here $\lambda_n := (n\pi)^2/4$ are the eigenvalues of the operator A defined on

$$D(A) := H^2 \cap H_0^1(I;\mathbb{C})$$

by

$$A\varphi := -\varphi'',$$

and the functions φ_n are the associated eigenvectors,

$$\varphi_n(x) := \sin(n\pi x/2),\ n \in \mathbb{N}^*, \text{ if } n \text{ is even}, \tag{19}$$

$$\varphi_n(x) := \cos(n\pi x/2),\ n \in \mathbb{N}^*, \text{ if } n \text{ is odd}. \tag{20}$$

Let us also introduce the following notations.

$$\mathbb{S} := \{\varphi \in L^2(I;\mathbb{C}); \|\varphi\|_{L^2(I;\mathbb{C})} = 1\},$$

$$H^7_{(0)}(I;\mathbb{C}) := \{\varphi \in H^7(I;\mathbb{C}); \varphi^{(2k)}(0) = \varphi^{(2k)}(1) = 0 \text{ for } k = 0,1,2,3\}.$$

With these notations, one has the following result, proved in [2].

Theorem 4 *For every $n \in \mathbb{N}^*$, there exists $\eta_n > 0$ such that, for every $n_0, n_f \in \mathbb{N}^*$, for every $(\psi^0, S^0, D^0), (\psi^1, S^1, D^1) \in [\mathbb{S} \cap H^7_{(0)}(I;\mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi^0 - \varphi_{n_0}\|_{H^7} + |S^0| + |D^0| < \eta_{n_0}, \tag{21}$$

$$\|\psi^1 - \varphi_{n_f}\|_{H^7} + |S^1| + |D^1| < \eta_{n_f}, \tag{22}$$

there exist a time $T > 0$ and (ψ, S, D, u) such that

$$\Psi \in C^0([0,T]; H^2 \cap H_0^1(I;\mathbb{C})) \cap C^1([0,T]; L^2(I;\mathbb{C})), \tag{23}$$

$$u \in H_0^1((0,T),\mathbb{R}), \tag{24}$$

$$S \in C^1([0,T];\mathbb{R}),\ D \in C^2([0,T];\mathbb{R}), \tag{25}$$

$$(15),\ (16) \text{ and } (17) \text{ hold}, \tag{26}$$

$$(\psi(0), S(0), D(0)) = (\psi^0, S^0, D^0), \tag{27}$$

$$(\psi(T), S(T), D(T)) = (\psi^1, S^1, D^1). \tag{28}$$

Thus, we also have the following corollary.

Corollary 5 *For every $n_0, n_f \in \mathbb{N}^*$, there exist a time $T > 0$ and (ψ, S, D, u) satisfying* (23)-(24)-(25)-(26) *such that* $(\psi(0), S(0), D(0)) = (\varphi_{n_0}, 0, 0)$, $(\psi(T), S(T), D(T)) = (\varphi_{n_f}, 0, 0)$.

Remark 6 *Let us point out that, if one does not care of S and D and if $(n_0, n_f) = (1,1)$, Theorem 4 is due to Karine Beauchard [1].*

References

[1] Karine Beauchard. *Local controllability of a 1-D Schrödinger equation. J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.

[2] Karine Beauchard and Jean-Michel Coron. *Controllability of a quantum particle in a moving potential well. J. Functional Analysis*, 232:328–389, 2006.

[3] Jean-Michel Coron. *Global asymptotic stabilization for controllable systems without drift. Math. Control Signals Systems*, 5(3):295–312, 1992.

[4] Jean-Michel Coron. *Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. C. R. Acad. Sci. Paris Sér. I Math.*, 317(3):271–276, 1993.

[5] Jean-Michel Coron. *On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. ESAIM Control Optim. Calc. Var.*, 1:35–75 (electronic), 1995/96.

[6] Jean-Michel Coron. *On the controllability of 2-D incompressible perfect fluids. J. Math. Pures Appl. (9)*, 75(2):155–188, 1996.

[7] Jean-Michel Coron. *Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. ESAIM Control Optim. Calc. Var.*, 8:513–554, 2002. A tribute to J. L. Lions.

[8] Jean-Michel Coron and Emmanuelle Crépeau. *Exact boundary controllability of a nonlinear KdV equation with critical lengths. J. Eur. Math. Soc. (JEMS)*, 6(3):367–398, 2004.

[9] Jean-Michel Coron and Andrei V. Fursikov. *Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. Russian J. Math. Phys.*, 4(4):429–448, 1996.

[10] Jean-Michel Coron and Emmanuel Trélat. *Global steady-state controllability of one-dimensional semilinear heat equations. SIAM J. Control Optim.*, 43(2):549–569 (electronic), 2004.

[11] Jean-Michel Coron and Emmanuel Trélat. *Global steady-state stabilization and controllability of 1-D semilinear wave equations. Contemporary Mathematics, to appear*, 2005.

[12] Lokenath Debnath. *Nonlinear water waves.* Academic Press Inc., Boston, MA, 1994.

[13] François Dubois, Nicolas Petit, and Pierre Rouchon. *Motion planning and nonlinear simulations for a tank containing a fluid. ECC*, 1999.

[14] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Exact controllability of the Navier-Stokes and Boussinesq equations. Russian Math. Surveys*, 54:565–618, 1999.

[15] Olivier Glass. *Exact boundary controllability of 3-D Euler equation. ESAIM Control Optim. Calc. Var.*, 5:1–44 (electronic), 2000.

[16] Olivier Glass. *On the controllability of the Vlasov-Poisson system. J. Differential Equations*, 195(2):332–379, 2003.

[17] Olivier Glass. *On the controllabilty of the 1-D isentropic Euler equation. to appear in J. Eur. Math. Soc. (JEMS)*, 2006.

[18] Thierry Horsin. *On the controllability of the Burgers equation. ESAIM Control Optim. Calc. Var.*, 3:83–95 (electronic), 1998.

[19] Pierre Rouchon. *Control of a quantum particule in a moving potential well. 2nd IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Seville*, 2003.

[20] Adhémar Jean Claude Barré de Saint-Venant. *Théorie du mouvement non permanent des eaux, avec applications aux crues des rivières et à l'introduction des marées dans leur lit. C. R. Acad. Sci. Paris Sér. I Math.*, 53:147–154, 1871.

[21] Michael Schmidt and Emmanuel Trélat. *Controllability of Couette flows. Commun. Pure Appl. Anal.*, 5(1):201–211, 2006.

Jean-Michel CORON – `Jean-Michel.Coron@math.u-psud.fr`
Université de Paris-Sud and Institut universitaire de France, Département de Mathématique, 91405 Orsay, France