Numerical Method for Elliptic Multiscale Problems

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A large class of multiscale problems are described by partial differential equations with highly oscillatory coefficients. Such coefficients represent the properties of a composite material or the heterogeneity of the medium in the computation of flow in porous media problem. The computation of an accurate discrete solution of such problems requires a very fine discretisation associated with a fine grid $\mathcal{T}_h$. For such a fine resolution, the storage and computation costs are very high. From an engineer's perspective, we are interested in the average behaviour of the elliptic oscillatory operator on a coarse scale taking into account the small scale features without fully resolving them. We intend to provide a smoother elliptic operator which on a coarse mesh behaves like the original operator. As a model problem, let us consider the elliptic boundary value problem on Ω, a bounded Lipschitz domain in $\mathbb{R}^d$,

$$\begin{cases} Lu = f\ , & \text{in } \Omega, \\ u = 0\ , & \text{on } \partial\Omega, \end{cases}$$

with the right-hand side f in $L^2(\Omega)$. As an example, let

$$L = -\sum_{i,j=1}^{d} \frac{\partial}{\partial_j} \alpha_{ij} \frac{\partial}{\partial_i},$$

whose coefficients may be non smooth, e.g., $\alpha_{ij} \in L^\infty(\Omega)$ is an oscillatory or jumping coefficient. We require certain real numbers $\underline{\lambda}, \overline{\lambda} > 0$ such that the matrix function $\alpha(x) = (\alpha_{ij})_{i,j=1,\dots,d}$ satisfies $0 < \underline{\lambda} \leq \lambda(\alpha(x)) \leq \overline{\lambda}$ for all eigenvalues λ of $\alpha(x)$ and almost all $x \in \Omega$. Note that there is no requirement on smoothness or periodicity of the coefficients. To simplify the theory and the numerical implementations, we restrict ourselves to the one-dimensional case. Our goal is to construct an elliptic operator A_0 with slowly varying coefficients which behaves similarly to the operator L on a coarse grid. To build A_0, we will consider the prolongation and restriction operators issued from the multi-grid method framework, and combine them with L. In the case of a T-periodic coefficient α, the homogenisation theory should provide a good operator [1]. Numerical results for different types of coefficient α (periodic or not) demonstrate the choice of the operator A_0 we built.

References

[1] Bensoussan, A. and Lions, J-L. and Papanicolaou, G., *Asymptotic analysis for periodic structures*, North-Holland Publishing Co., 1978.

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