On a free boundary shallow water model for tsunami simulation

Fabien FLORI, UMR CNRS 6134
Catherine GIACOMONI, UMR CNRS 6134

Keywords: tsunami, shallow water, free boundary, existence

We study the formation of tsunamis and the prediction of the sea progression inland when the wave touches the shore. We propose a formal survey of a model based on a time dependent fluid domain interacting with the lithosphere. We give sufficient conditions in order to show an existence result. This work clearly improves a first study [1] in which the domain was considered fixed.

At time $t$, the water surface occupies a bounded domain $\Omega_t$ of $\mathbb{R}^2$ with boundary $\gamma_t$. We denote by $\gamma_0$ the boundary of the fluid at initial time. Assuming that $\gamma_0$ is smooth enough, we define the deformed boundary as follows: $\gamma_t := \{ x = X + d(X, t), X \in \gamma_0 \}$, where $d$ corresponds to the displacement $d(X, t) = \Gamma(t, 0, X) - X$ and $\Gamma(t, s, x)$ denotes the Lagrangian flow. This deformation has a meaning if the corresponding Lagrangian flow $X \mapsto \Gamma(t, 0, X)$ is $X + d(X, t)$ is a diffeomorphism from $\gamma_0$ onto $\gamma_t := \Gamma(t, 0, \gamma_0)$, so that all what follows will hold as long as $\det \Gamma(\cdot, 0, \cdot)$ and $\Gamma(t, s, x)$ is one-to-one on $\gamma_0$. Thus we define $\Gamma(0, t, x)$ by $\Gamma(0, t, x) := \Gamma(t, 0, x)^{-1}$ and $\Gamma(t, s, x) = \Gamma(t, 0, \Gamma(0, s, x))$.

If we consider small data, our model allows to bound $d$ in $W^{1,\infty}(0; W^{1,\infty}(\gamma_0))$ and thus to show that the deformation has a meaning. We denote by $h(x, t)$ the total height of the water column. We set $h(x, t) = H + \zeta(x, t) - U(x, t)$ where $H$ represents the constant depth of the domain and $\zeta(x, t)$ the surface elevation. We notice $Q = \cup_{t \in (0, T)} \Omega_t \times \{ t \}$, $Q_0 = \Omega_0 \times (0, T)$, $\Sigma = \cup_{t \in (0, T)} \gamma_t \times \{ t \}$, $\Sigma_0 = \gamma_0 \times (0, T)$ and $n$ the exterior unit normal to $\Omega_t$ on $\gamma_t$. The motion of the lithosphere can be modeled by a thin plate operator $\sigma_{\gamma} U = \Delta_{\gamma} U + \Delta U = f - gh$ in $Q_0$. The action of the water on the plate is taken into account by a term of pressure $-gh$ where $h = \zeta + H - U$. Last, let us suppose that the model is balanced at $t = 0$ from where $\int_0^t f_g$ is given by $f = g(\zeta_0 + H - U_0)$. Since the most significant tsunamis take place near the coastline, we suppose that the fluid is governed by the following shallow water problem

\begin{equation}
(F) \begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - A \Delta u + g \nabla \zeta + \omega_c \wedge u + C_d |u| u = f' & \text{in } Q \\
\frac{\partial h}{\partial t} + \text{div}(uh) = 0, & \text{in } Q
\end{cases}
\end{equation}

where $u$ is the velocity, $\omega_c \wedge u$ is the Coriolis term, $C_d |u| u$ is the bottom shear and $f'$ is the action of the wind on the water surface. Let us note that this model naturally takes into account the variations of the lithosphere. In order to set the boundary conditions, we introduce the Lagrangian description of the velocity, $U : \gamma_0 \times (0, T) \rightarrow \mathbb{R}^2$, $(X, T) \mapsto u(\Gamma(t, 0, X), t)$. On boundary $\gamma_0$ we have $U(X, t) = u(X + d(X, t), t) = \frac{\partial d(X, t)}{\partial t}$, and we characterize the boundary motion $\gamma_1$ by a condition on the fluid stress tensor $\sigma$

\[ \sigma(X + d(X, t), t) \cdot n(X + d(X, t), t) | \det J(X, t) = A(\partial U(X, t)/\partial t) | \text{on } \gamma_0 \times (0, T) \]

where $A$ is an operator defined on $\gamma_0$ which takes into account the stress applied to the fluid on the boundary when the water invades land. We assume that $A$ is a Laplace-Beltrami operator which ensures that $\int_{\gamma_0} A(v) v = \int_{\gamma_0} A^{1/2}(v) A^{1/2}(v) = |v|^2_{H^2(\gamma_0)}$.

References


Fabien FLORI – flori@univ-corse.fr
UMR CNRS 6134, Université de Corse, BP 52, 20250 Corte
Catherine GIACOMONI – giaco@univ-corse.fr
UMR CNRS 6134, Université de Corse, BP 52, 20250 Corte