On a free boundary shallow water model for tsunami simulation

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We study the formation of tsunamis and the prediction of the sea progression inland when the wave touches the shore. We propose a formal survey of a model based on a time dependent fluid domain interacting with the lithosphere. We give sufficient conditions in order to show an existence result. This work clearly improves a first study [1] in which the domain was considered fixed.

At time t, the water surface occupies a bounded domain Ω_t of $\mathbb{R}^2$ with boundary γ_t. We denote by γ_0 the boundary of the fluid at initial time. Assuming that γ_0 is smooth enough, we define the deformed boundary as follows: $\gamma_t := \{x = X + d(X,t),\ X \in \gamma_0\}$, where d corresponds to the displacement $d(X,t) = \Gamma(t,0,X) - X$ and $\Gamma(t,s,x)$ denotes the Lagrangian flow. This deformation has a meaning if the corresponding Lagrangian flow $X \mapsto \Gamma(t,0,X) = X + d(X,t)$ is a diffeomorphism from γ_0 onto $\gamma_t := \Gamma(t,0,\gamma_0)$, so that all what follows will hold as long as $\det J(X,t) \neq 0$ on γ_0, (where $J(X,t)$ is the Jacobian matrix associated to the transformation $X \mapsto \Gamma(t,0,X)$), and Γ is one-to-one on γ_0. Thus we define $\Gamma(0,t,x)$ by $\Gamma(0,t,.) = \Gamma(t,0,.)^{-1}$ and $\Gamma(t,s,x) = \Gamma(t,0,\Gamma(0,s,x))$.

If we consider small data, our model allows to bound d in $W^{1,\infty}(0,T;W^{1,\infty}(\gamma_0))$ and thus to show that the deformation has a meaning. We denote by $h(x,t)$ the total height of the water column. We set $h(x,t) = H + \zeta(x,t) - U(x,t)$ where H represents the constant depth of the domain and $\zeta(x,t)$ the surface elevation. We notice $Q = \cup_{t\in(0,T)}\Omega_t \times \{t\}$, $Q_0 = \Omega_0 \times (0,T)$, $\Sigma = \cup_{t\in(0,T)}\gamma_t \times \{t\}$, $\Sigma_0 = \gamma_0 \times (0,T)$ and n the exterior unit normal to Ω_t on γ_t. The motion of the lithosphere can be modeled by a thin plate operator $\frac{\partial^2 U}{\partial t^2} - \Delta\frac{\partial^2 U}{\partial t^2} + \Delta^2 U = f - gh$ in Q_0. The action of the water on the plate is taken into account by a term of pressure $-gh$ where $h = \zeta + H - U$. Last, let us suppose that the model is balanced at $t = 0$ from where f_0 is given by $f = g(\zeta_0 + H - U_0)$. Since the most significant tsunamis take place near the coastline, we suppose that the fluid is governed by the following shallow water problem

$$(F)\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - A\Delta u + g\nabla\zeta + \omega_c \wedge u + C_d|u|u = f' & \text{in } Q \\ \dfrac{\partial h}{\partial t} + \operatorname{div}(uh) = 0, & \text{in } Q \end{cases}$$

where u is the velocity, $\omega_c \wedge u$ is the Coriolis term, $C_d|u|u$ is the bottom shear and f' is the action of the wind on the water surface. Let us note that this model naturally takes into account the variations of the lithosphere. In order to set the boundary conditions, we introduce the Lagrangian description of the velocity, $U : \gamma_0 \times (0,T) \to \mathbb{R}^2$, $(X,T) \mapsto u(\Gamma(t,0,X),t)$. On boundary γ_0 we have $U(X,t) = u(X + d(X,t),t) = \frac{\partial d(X,t)}{\partial t}$, and we characterize the boundary motion γ_t by a condition on the fluid stress tensor σ

$$\sigma(X + d(X,t),t)\cdot n(X + d(X,t),t)|\det J|(X,t) = A(\partial U(X,t)/\partial t) \text{ on } \gamma_0 \times (0,T)$$

where A is an operator defined on γ_0 which takes into account the stress applied to the fluid on the boundary when the water invades land. We assume that A is a Laplace-Beltrami operator which ensures that $\int_{\gamma_0} A(v)v = \int_{\gamma_0} A^{1/2}(v)A^{1/2}(v) = \|v\|^2_{H^2(\gamma_0)}$.

References

[1] Bernard Di Martino, Fabien Flori, Catherine Giacomoni, Pierre Orenga, *Mathematical and numerical analysis of a tsunami problem*, M3AS, Vol. 13, No.10, 1489-1514, 2003.

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