Discontinuous Galerkin Methods as Weighted Residuals

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We present a very recent point of view on Discontinuous Galerkin Methods where the formulation is seen as a weighted residual method.

Let us see the basic ideas behind it. We consider a rather general system of PDE equations in a domain Ω and on its boundary $\partial\Omega$, that we write as

$$\mathcal{A}(u) = 0. \tag{1}$$

In (1) the (nonlinear) operator $\mathcal{A}$ includes the right-hand side as well as the boundary conditions, so that, say, problem $\Delta u = f$ in Ω, with $u = g$ on the boundary $\partial\Omega$, would correspond to the operator $\mathcal{A}(u) := (\Delta u - f, u_{|\partial\Omega} - g)$ in suitable functional spaces. The case of a vector valued function u is obviously included, but in the following discussion we will always talk about *functions*, without specifying whether they are scalar or vector valued.

We now take a decomposition of the domain Ω into elements, and a piecewise smooth function w (possibly discontinuous from one element to another of the decomposition) and we want to write that w is a solution of (1). In order to do that, we have to take into account the residual of the partial differential equation(s) inside each element, the residual in the boundary conditions on $\partial\Omega$, *and* the jumps of suitable trace operators applied to w at the interfaces from one element to another (that we consider as *residuals* of suitable continuity conditions). We indicate all these residuals by $R_1(w), R_2(w), ..., R_n(w)$.

We choose now a finite element space V_h of piecewise polynomial functions, and we would like to require, in some discrete sense, that $R_i(u_h) = 0, \quad i = 1, 2, ..., n$ (where u_h is obviously the discrete solution we are looking for). For this we could, in principle, weight the residuals as in a least squares method. However it is more general and more convenient to define suitable corresponding weight-operators $W_1(v_h), W_2(v_h), ..., W_n(v_h)$ acting on the test functions v_h, and write the corresponding discrete problem as: *Find $u_h \in V_h$ such that*

$$\sum_{i=1}^{n} (R_i(u_h), W_i(v_h))_i = 0 \qquad \forall\, v_h \in V_h, \tag{2}$$

where the $(.\,,\,.)_i$'s are suitable (in general, L^2-type) inner products (either inside each element, or on the boundaries of the elements). The formulation (2) is quite general and it applies to a number of different problems, but for the sake of simplicity the lecture will discuss only the case of linear elliptic operators.

The stability and the accuracy of the resulting scheme depend heavily on the choice of the weights $W_1(v_h), W_2(v_h), ..., W_n(v_h)$. Suitable choices of the weights $W_1(v_h), W_2(v_h), ..., W_n(v_h)$ can help in getting a *nicer* formulation (e.g. symmetric, when the problem is itself symmetric, easier to implement, and so on). But in order to reach a stable method it is, in general, necessary to add *a least square part* to them. Roughly speaking, this means choosing at least one of the W_i of the form

$$W_i = N_i + \gamma_i R_i \tag{3}$$

where N_i (possibly equal to zero) is the part used to make the method nicer, γ_i is a suitable (in general, mesh dependent) coefficient, and the term $\gamma_i R_i$ is there to *stabilize* the method. It is surprising to see ([1], [2]) that, in a number of cases, internal stabilizations à la Hughes-Franca and stabilizations on the interelement jump terms (much more common in the DG practice) produce the same stabilizing effects, and sometimes even the same scheme.

References

[1] F. Brezzi, T.J.R. Hughes, L.D. Marini, and A. Masud, *Mixed Discontinuous Galerkin methods for Darcy flow*, J. Sci. Comput. **22** (2005), 119-145.

[2] F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli *Stabilization mechanisms in discontinuous Galerkin finite element methods*, to appear in Comp. Meth. Appl. Mech. Engrg.

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