Two discrete inequalities of Poincaré-Friedrichs in discontinuous spaces for Maxwell's equations

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Abstract

We present two new discrete inequalities of Poincaré-Friedrichs on discontinuous spaces for Maxwell's equations. The proofs of the inequalities are based on some decompositions formulas of $L^2(\Omega)^3$.

1 Some notations and spaces

Throughout this paper, Ω will denote a bounded Lipschitz polyhedron included in $\mathbb{R}^3$ which is supposed to be both connected and simply connected. Γ is the boundary of Ω which is also assumed to be connected and simply connected. Given a domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$, we denote by $H^s(D)^d$, $d = 1, 2, 3$, the Sobolev space of real valued functions with integer or fractional regulatity exponent $s \geqslant 0$, endowed with the norm $\| \cdot \|_{s,D}$; see, e.g, [5]. For $D \subset \mathbb{R}^3$, $H(\nabla\times, D)$ and $H(\nabla\cdot, D)$ are the spaces of real valued vector functions $u \in L^2(D)^3$ with $\nabla\times u \in L^2(D)^3$ and $\nabla\cdot u \in L^2(D)$, respectively, endowed with the graph norms. We denote by $H_0^1(D),\ H_0(\nabla\times, D),\ H_0(\nabla\cdot, D)$ the subspaces of $H^1(D),\ H(\nabla\times, D),\ H(\nabla\cdot, D)$ of functions with zero trace, tangential trace and normal trace on ∂D, respectively. The spaces $H(\nabla\times^0, D)$ and $H(\nabla\cdot^0, D)$ are the subspaces of $H(\nabla\times, D)$ and $H(\nabla\cdot, D)$ consisting of irrotational and $divergence-$free functions, respectively. We denote by $H_0^1(D),\ H_0(\nabla\times, D),\ H_0(\nabla\cdot, D)$ the subspaces of $H^1(D),\ H(\nabla\times, D),\ H(\nabla\cdot, D)$ of functions with zero trace, tangential trace and normal trace on ∂D, respectively . We assume that Ω satisfies $H_0(\nabla\times, \Omega) \cap H(\nabla\cdot, \Omega)$ and $H(\nabla\times, \Omega) \cap H_0(\nabla\cdot, \Omega)$ are both continuously imbedded in $H^1(\Omega)^3$.
Let Π_h be a partition into tetrahedra for Ω. If K in Π_h we denote by h_K the diameter

of K and set $h := max_{K \in \Pi_h} h_K$.

Faces: We define and characterise the faces of the triangulation Π_h. An interior face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K_1 \cap \partial K_2$, where K_1 and K_2 are two adjacent elements of Π_h. A boundary face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K \cap \partial \Omega$, where K is a boundary element of Π_h. We denote by F_h^I the union of all interior faces of Π_h, by F_h^D the union of all boundary faces of Π_h and let F_h denote the union of all faces of Π_h. Furthermore we associate F_h^D with Γ since Ω is a polyhedron.

Traces: Let $H^s(\Pi_h) = \{v \ : \ v_{|K} \in H^s(K) \ \ \forall K \in \Pi_h\}$ for $s > \frac{1}{2}$ be endowed with the norm $\|v\|^2_{s,\Pi_h} = \sum_{K \in \Pi_h} \|v\|^2_{s,K}$. Then the elementwise traces of functions in $H^s(\Pi_h)$ belong to $TR(F_h) = \Pi_{K \in \Pi_h} L^2(\partial K)$; they are double-valued on F_h^I and single-valued on F_h^D. The space $L^2(F_h)$ can be identified with the functions in $TR(F_h)$ for which the two traces values coincide.
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Trace operators: Let us introduce the following trace operators for piecewise smooth functions. First, let $w \in TR(F_h)^3$ and $e \in F_h$. If e is an interior face in F_h^I, we denote by K_1 and K_2 the elements sharing e, by n_i the normal unit vector pointing exterior to K_i and we set $\omega_i = \omega_{|\partial K_i}, i = 1, 2$. We define the $average$, $tangential$ and $normal\ jumps$ of w at $x \in e$ as

$$\{\omega\} = \frac{\omega_1 + \omega_2}{2} \ , \ \ [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2 \ \text{ and } \ [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2.$$

If $e \in F_h^D$, we set for $x \in e$

$$\{\omega\} = \omega \ , \ \ [\omega]_T = n \times \omega \ \text{ and } \ [\omega]_N = n \cdot \omega.$$

We denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)^3$ or $L^2(\Omega)$ and by $\| \cdot \| = \| \cdot \|_{0,\Omega} = \| \cdot \|_{L^2(\Omega)^3}$ or $\| \cdot \|_{L^2(\Omega)}$. For $e \in F_h$, we denote by $< \cdot, \cdot >_e$ the scaler product in $L^2(e)^3$ or $L^2(e)$. Furthermore if F_h^D is identified to $\partial \Omega$, we identify $\sum_{e \in F_h^D} < \cdot, \cdot >_e$ to $< \cdot, \cdot >$, the scalar product in $L^2(\partial \Omega)^3$ or $L^2(\partial \Omega)$. In the previous notation we can

state the basic integration by parts formulas
$\forall v, u \in H^1(\Pi_h)^3$, $\forall \psi \in H^1(\Pi_h)$, we have

$$\begin{aligned}(\nabla \times u, v) = (u, \nabla \times v) \quad &+ \quad < n \times u, v > \\ &+ \quad \sum_{e \in F_h^I} < [u]_T, \{v\} >_e - < [v]_T, \{u\} >_e \end{aligned} \qquad (1)$$

and

$$\begin{aligned}(\nabla \cdot u, \psi) = (u, \nabla \psi) \quad &+ \quad < u \cdot n, \psi > \\ &+ \quad \sum_{e \in F_h^I} < [u]_N, \{\psi\} >_e + < [\psi], \{u\} \cdot n >_e . \end{aligned} \qquad (2)$$

2 The first inequality

Lemma 2.1 *Let $u \in H^1(\Pi_h)^3$ and let $\sigma = \kappa \frac{p^2}{h}$ with $p \geqslant 1$ is an integer. Then, there exists a constant C independent of h and p such that*

$$\|u\|^2 \leqslant C(\|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2).$$

Proof: Let us first denote that the following orthogonal decomposition formula holds if $\partial\Omega$ is simply-connected (see [2])

$$L^2(\Omega)^3 = H_0(\nabla \times 0, \Omega) \oplus H(\nabla \cdot 0, \Omega).$$

Now, let $u \in H^1(\Pi_h)^3$, then $u \in L^2(\Omega)^3$ and we can decompose u as

$$u = u_1 + u_2 \text{ with } u_1 \in H_0(\nabla \times 0, \Omega) \text{ and } \; u_2 \in H(\nabla \cdot 0, \Omega). \qquad (3)$$

As in [2], we show that $u_1 \in H_0(\nabla \times 0, \Omega)$ if and only if $u_1 = \nabla q$ with $q \in H_0^1(\Omega)$. We also show that $u_2 = \nabla \times \phi$ with $\phi \in H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$ and such that $u_2 = \nabla \times \phi$ in Ω. In particular, the traces of ϕ are well defined since $\phi \in H_0(\nabla \times, \Omega) \cap H(\nabla \cdot, \Omega) \hookrightarrow H^1(\Omega)^3$. Note that (3) imply

$$\begin{aligned}\|u\|^2 &= (\nabla q + \nabla \times \phi, \nabla q + \nabla \times \phi) \\ &= (\nabla q, \nabla q) + (\nabla \times \phi, \nabla \times \phi) \\ &= \|\nabla q\|^2 + \|\nabla \times \phi\|^2.\end{aligned}$$

Now, by using (1) and (2), we obtain

$$\begin{aligned}\|u\|^2 &= (u, \nabla q) + (u, \nabla \times \phi) \\ &= -(\nabla \cdot u, q) + (\nabla \times u, \phi) + \sum_{e \in F_h^I} < [u]_N, q >_e - < [u]_T, \phi >_e \\ &\quad + \sum_{e \in F_h^D} < u \cdot n, q >_e + < n \times u, \phi >_e.\end{aligned}$$

Then, since q is in $H_0^1(\Omega)$,

$$\begin{aligned}\|u\|^2 = -(\nabla\cdot u, q) + (\nabla\times u, \phi) \quad &+ \quad \sum_{e\in F_h^I} < [u]_N, q >_e - < [u]_T, \phi >_e \\ &+ \quad \sum_{e\in F_h^D} < n\times u, \phi >_e.\end{aligned}$$

So

$$\begin{aligned}\|u\|^2 \leqslant C(\|\nabla\cdot u\|^2 \quad &+ \quad \|\nabla\times u\|^2 + \sum_{e\in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2 + \sum_{e\in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2)^{\frac{1}{2}} \\ &\times (\|q\|^2 + \|\phi\|^2 + \sum_{e\in F_h^I} \|\frac{1}{\sqrt{\sigma}} q\|_{0,e}^2 + \sum_{e\in F_h} \|\frac{1}{\sqrt{\sigma}}\phi\|_{0,e}^2)^{\frac{1}{2}}.\end{aligned}$$

It is clear that

$$\|q\|^2 \leqslant C(\Omega)\|\nabla q\|^2 \leqslant C(\Omega)\|u\|^2.$$

Since $\phi \in H(\nabla\times, \Omega)\cap H_0(\nabla\cdot, \Omega)$ and $\nabla\cdot\phi = 0$, we obtain (see [2] for the first inequality)

$$\begin{aligned}\|\phi\|^2 \quad &\leqslant \quad C(\Omega)(\|\nabla\times\phi\|^2 + \|\nabla\cdot\phi\|^2) \\ &\leqslant \quad C(\Omega)\|\nabla\times\phi\|^2 \\ &\leqslant \quad C(\Omega)\|u\|^2\end{aligned}$$

Now, by using trace inequality (see [6]), we have for any $e \in F_h$

$$\begin{aligned}\|\frac{1}{\sqrt{\sigma}} q\|_{0,e}^2 \quad &\leqslant \quad \frac{C}{\sigma}(\frac{1}{h_K}\|q\|_{0,K}^2 + \|q\|_{0,K}\|\nabla q\|_{0,K}) \\ &\leqslant \quad C\frac{h}{p^2}(\frac{1}{h_K}\|q\|_{0,sK}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + h_K\|\nabla q\|_{0,K}^2) \\ &\leqslant \quad C\frac{h}{p^2}(\frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|\nabla q\|_{0,K}^2) \\ &\leqslant \quad C(\|q\|_{0,K}^2 + \|\nabla q\|_{0,K}^2).\end{aligned}$$

In particular

$$\begin{aligned}\sum_{e\in F_h^I} \|\frac{1}{\sqrt{\sigma}} q\|_{0,e}^2 \quad &\leqslant \quad C\sum_{K\in\Pi_h} \|q\|_{0,K}^2 + \|\nabla q\|_{0,K}^2 \\ &\leqslant \quad C(\|q\|^2 + \|\nabla q\|^2) \\ &\leqslant \quad C\|u\|^2\end{aligned}$$

In the same manner, using the imbedding of $H(\nabla\times, \Omega)\cap H_0(\nabla\cdot, \Omega)$ in $H^1(\Omega)^3$; we can bound $\sum_{e\in F_h} \|\frac{1}{\sqrt{\sigma}}\phi\|_{0,e}^2$ and obtain

$$\begin{aligned}\|\frac{1}{\sqrt{\sigma}}\phi\|_{0,F_h}^2 \leqslant C\|\phi\|_{1,\Omega}^2 \leqslant C\|\phi\|_{H(\nabla\times,\Omega)\cap H_0(\nabla\cdot,\Omega)}^2 \quad &\leqslant \quad C(\|\nabla\times\phi\|^2 + \|\nabla\cdot\phi\|^2) \\ &\leqslant \quad C\|\nabla\times\phi\|^2 \\ &\leqslant \quad C\|u\|^2.\end{aligned}$$

Finally, we obtain

$$\|u\|^2 \leqslant C(\|\nabla\cdot u\|^2 + \|\nabla\times u\|^2 + \sum_{e\in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2 + \sum_{e\in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2)^{\frac{1}{2}}\|u\|. \quad \square$$

3 The second inequality

Lemma 3.1 *Let $u \in H^1(\Pi_h)^3$ and let $\sigma = \kappa \frac{p^2}{h}$ with $p \geqslant 1$ is an integer. Then, there exists C independent of h and p such that*

$$\|u\|^2 \leqslant C(\|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_N\|_{0,e}^2).$$

Proof: The proof is similar to the proof in the previous section. But here we use the following orthogonal decomposition formula if Ω is simply-connected (see also [2], [4])

$$L^2(\Omega)^3 = H(\nabla \times 0, \Omega) \oplus H_0(\nabla \cdot 0, \Omega).$$

Then, for $u \in L^2(\Omega)^3$ we write

$$u = u_1 + u_2$$

with $u_1 \in H(\nabla \times 0, \Omega)$ and $u_2 \in H_0(\nabla \cdot 0, \Omega)$. Since $\nabla \times u_1 = 0$, we write $u_1 = \nabla q$ with $q \in H^1(\Omega)$ and since $u_2 \in H_0(\nabla \cdot 0, \Omega)$, we write $u_2 = \nabla \times \varphi$ with $\varphi \in H_0(\nabla \times, \Omega) \cap H(\nabla \cdot 0, \Omega)$ (see [2], [4]).

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