

# Two discrete inequalities of Poincaré-Friedrichs in discontinuous spaces for Maxwell's equations

Daveau C<sup>‡</sup> and Zaghdani A<sup>†</sup>

<sup>†</sup> Université d'Orsay,

Laboratoire de Mathématiques,

Equipe d'Analyse Numérique,

91405 Orsay, France.

<sup>‡</sup> Laboratoire d'Analyse, Géométrie, et Modélisation,

Département de mathématiques,

U.M.R. au C.N.R.S 8088,

Université de Cergy-Pontoise.

## Abstract

We present two new discrete inequalities of Poincaré-Friedrichs on discontinuous spaces for Maxwell's equations. The proofs of the inequalities are based on some decompositions formulas of  $L^2(\Omega)^3$ .

## 1 Some notations and spaces

Throughout this paper,  $\Omega$  will denote a bounded Lipschitz polyhedron included in  $\mathbf{R}^3$  which is supposed to be both connected and simply connected.  $\Gamma$  is the boundary of  $\Omega$  which is also assumed to be connected and simply connected. Given a domain  $D$  in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ , we denote by  $H^s(D)^d$ ,  $d = 1, 2, 3$ , the Sobolev space of real valued functions with integer or fractional regularity exponent  $s \geq 0$ , endowed with the norm  $\|\cdot\|_{s,D}$ ; see, e.g. [5]. For  $D \subset \mathbf{R}^3$ ,  $H(\nabla \times, D)$  and  $H(\nabla \cdot, D)$  are the spaces of real valued vector functions  $u \in L^2(D)^3$  with  $\nabla \times u \in L^2(D)^3$  and  $\nabla \cdot u \in L^2(D)$ , respectively, endowed with the graph norms. We denote by  $H_0^1(D)$ ,  $H_0(\nabla \times, D)$ ,  $H_0(\nabla \cdot, D)$  the subspaces of  $H^1(D)$ ,  $H(\nabla \times, D)$ ,  $H(\nabla \cdot, D)$  of functions with zero trace, tangential trace and normal trace on  $\partial D$ , respectively. The spaces  $H(\nabla \times^0, D)$  and  $H(\nabla \cdot^0, D)$  are the subspaces of  $H(\nabla \times, D)$  and  $H(\nabla \cdot, D)$  consisting of irrotational and *divergence*-free functions, respectively. We denote by  $H_0^1(D)$ ,  $H_0(\nabla \times, D)$ ,  $H_0(\nabla \cdot, D)$  the subspaces of  $H^1(D)$ ,  $H(\nabla \times, D)$ ,  $H(\nabla \cdot, D)$  of functions with zero trace, tangential trace and normal trace on  $\partial D$ , respectively. We assume that  $\Omega$  satisfies  $H_0(\nabla \times, \Omega) \cap H(\nabla \cdot, \Omega)$  and  $H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$  are both continuously imbedded in  $H^1(\Omega)^3$ .

Let  $\Pi_h$  be a partition into tetrahedra for  $\Omega$ . If  $K$  in  $\Pi_h$  we denote by  $h_K$  the diameter

of  $K$  and set  $h := \max_{K \in \Pi_h} h_K$ .

**Faces:** We define and characterise the faces of the triangulation  $\Pi_h$ . An interior face of  $\Pi_h$  is defined as the (non-empty) two-dimensional interior of  $\partial K_1 \cap \partial K_2$ , where  $K_1$  and  $K_2$  are two adjacent elements of  $\Pi_h$ . A boundary face of  $\Pi_h$  is defined as the (non-empty) two-dimensional interior of  $\partial K \cap \partial \Omega$ , where  $K$  is a boundary element of  $\Pi_h$ . We denote by  $F_h^I$  the union of all interior faces of  $\Pi_h$ , by  $F_h^D$  the union of all boundary faces of  $\Pi_h$  and let  $F_h$  denote the union of all faces of  $\Pi_h$ . Furthermore we associate  $F_h^D$  with  $\Gamma$  since  $\Omega$  is a polyhedron.

**Traces:** Let  $H^s(\Pi_h) = \{v : v|_K \in H^s(K) \ \forall K \in \Pi_h\}$  for  $s > \frac{1}{2}$  be endowed with the norm  $\|v\|_{s, \Pi_h}^2 = \sum_{K \in \Pi_h} \|v\|_{s, K}^2$ . Then the elementwise traces of functions in  $H^s(\Pi_h)$  belong to  $TR(F_h) = \prod_{K \in \Pi_h} L^2(\partial K)$ ; they are double-valued on  $F_h^I$  and single-valued on  $F_h^D$ . The space  $L^2(F_h)$  can be identified with the functions in  $TR(F_h)$  for which the two traces values coincide.

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**Trace operators:** Let us introduce the following trace operators for piecewise smooth functions. First, let  $w \in TR(F_h)^3$  and  $e \in F_h$ . If  $e$  is an interior face in  $F_h^I$ , we denote by  $K_1$  and  $K_2$  the elements sharing  $e$ , by  $n_i$  the normal unit vector pointing exterior to  $K_i$  and we set  $\omega_i = \omega|_{\partial K_i}$ ,  $i = 1, 2$ . We define the *average*, *tangential* and *normal jumps* of  $w$  at  $x \in e$  as

$$\{\omega\} = \frac{\omega_1 + \omega_2}{2}, \quad [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2 \quad \text{and} \quad [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2.$$

If  $e \in F_h^D$ , we set for  $x \in e$

$$\{\omega\} = \omega, \quad [\omega]_T = n \times \omega \quad \text{and} \quad [\omega]_N = n \cdot \omega.$$

We denote by  $(\cdot, \cdot)$  the scalar product in  $L^2(\Omega)^3$  or  $L^2(\Omega)$  and by  $\|\cdot\| = \|\cdot\|_{0, \Omega} = \|\cdot\|_{L^2(\Omega)^3}$  or  $\|\cdot\|_{L^2(\Omega)}$ . For  $e \in F_h$ , we denote by  $\langle \cdot, \cdot \rangle_e$  the scaler product in  $L^2(e)^3$  or  $L^2(e)$ . Furthermore if  $F_h^D$  is identified to  $\partial \Omega$ , we identify  $\sum_{e \in F_h^D} \langle \cdot, \cdot \rangle_e$  to  $\langle \cdot, \cdot \rangle$ , the scalar product in  $L^2(\partial \Omega)^3$  or  $L^2(\partial \Omega)$ . In the previous notation we can

state the basic integration by parts formulas  
 $\forall v, u \in H^1(\Pi_h)^3$ ,  $\forall \psi \in H^1(\Pi_h)$ , we have

$$\begin{aligned} (\nabla \times u, v) &= (u, \nabla \times v) + \langle n \times u, v \rangle \\ &+ \sum_{e \in F_h^I} \langle [u]_T, \{v\} \rangle_e - \langle [v]_T, \{u\} \rangle_e \end{aligned} \quad (1)$$

and

$$\begin{aligned} (\nabla \cdot u, \psi) &= (u, \nabla \psi) + \langle u \cdot n, \psi \rangle \\ &+ \sum_{e \in F_h^I} \langle [u]_N, \{\psi\} \rangle_e + \langle [\psi], \{u\} \cdot n \rangle_e. \end{aligned} \quad (2)$$

## 2 The first inequality

**Lemma 2.1** *Let  $u \in H^1(\Pi_h)^3$  and let  $\sigma = \kappa \frac{p^2}{h}$  with  $p \geq 1$  is an integer. Then, there exists a constant  $C$  independent of  $h$  and  $p$  such that*

$$\|u\|^2 \leq C(\|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2).$$

**Proof:** Let us first denote that the following orthogonal decomposition formula holds if  $\partial\Omega$  is simply-connected (see [2])

$$L^2(\Omega)^3 = H_0(\nabla \times 0, \Omega) \oplus H(\nabla \cdot 0, \Omega).$$

Now, let  $u \in H^1(\Pi_h)^3$ , then  $u \in L^2(\Omega)^3$  and we can decompose  $u$  as

$$u = u_1 + u_2 \text{ with } u_1 \in H_0(\nabla \times 0, \Omega) \text{ and } u_2 \in H(\nabla \cdot 0, \Omega). \quad (3)$$

As in [2], we show that  $u_1 \in H_0(\nabla \times 0, \Omega)$  if and only if  $u_1 = \nabla q$  with  $q \in H_0^1(\Omega)$ . We also show that  $u_2 = \nabla \times \phi$  with  $\phi \in H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$  and such that  $u_2 = \nabla \times \phi$  in  $\Omega$ . In particular, the traces of  $\phi$  are well defined since  $\phi \in H_0(\nabla \times, \Omega) \cap H(\nabla \cdot, \Omega) \hookrightarrow H^1(\Omega)^3$ . Note that (3) imply

$$\begin{aligned} \|u\|^2 &= (\nabla q + \nabla \times \phi, \nabla q + \nabla \times \phi) \\ &= (\nabla q, \nabla q) + (\nabla \times \phi, \nabla \times \phi) \\ &= \|\nabla q\|^2 + \|\nabla \times \phi\|^2. \end{aligned}$$

Now, by using (1) and (2), we obtain

$$\begin{aligned} \|u\|^2 &= (u, \nabla q) + (u, \nabla \times \phi) \\ &= -(\nabla \cdot u, q) + (\nabla \times u, \phi) + \sum_{e \in F_h^I} \langle [u]_N, q \rangle_e - \langle [u]_T, \phi \rangle_e \\ &+ \sum_{e \in F_h^D} \langle u \cdot n, q \rangle_e + \langle n \times u, \phi \rangle_e. \end{aligned}$$

Then, since  $q$  is in  $H_0^1(\Omega)$ ,

$$\begin{aligned} \|u\|^2 = & -(\nabla \cdot u, q) + (\nabla \times u, \phi) + \sum_{e \in F_h^I} \langle [u]_N, q \rangle_e - \langle [u]_T, \phi \rangle_e \\ & + \sum_{e \in F_h^D} \langle n \times u, \phi \rangle_e. \end{aligned}$$

So

$$\begin{aligned} \|u\|^2 \leq & C(\|\nabla \cdot u\|^2 + \|\nabla \times u\|^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2)^{\frac{1}{2}} \\ & \times (\|q\|^2 + \|\phi\|^2 + \sum_{e \in F_h^I} \|\frac{1}{\sqrt{\sigma}}q\|_{0,e}^2 + \sum_{e \in F_h} \|\frac{1}{\sqrt{\sigma}}\phi\|_{0,e}^2)^{\frac{1}{2}}. \end{aligned}$$

It is clear that

$$\|q\|^2 \leq C(\Omega)\|\nabla q\|^2 \leq C(\Omega)\|u\|^2.$$

Since  $\phi \in H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$  and  $\nabla \cdot \phi = 0$ , we obtain ( see [2] for the first inequality )

$$\begin{aligned} \|\phi\|^2 & \leq C(\Omega)(\|\nabla \times \phi\|^2 + \|\nabla \cdot \phi\|^2) \\ & \leq C(\Omega)\|\nabla \times \phi\|^2 \\ & \leq C(\Omega)\|u\|^2 \end{aligned}$$

Now, by using trace inequality (see [6] ), we have for any  $e \in F_h$

$$\begin{aligned} \|\frac{1}{\sqrt{\sigma}}q\|_{0,e}^2 & \leq \frac{C}{\sigma}(\frac{1}{h_K}\|q\|_{0,K}^2 + \|q\|_{0,K}\|\nabla q\|_{0,K}) \\ & \leq C\frac{h}{p^2}(\frac{1}{h_K}\|q\|_{0,sK}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + h_K\|\nabla q\|_{0,K}^2) \\ & \leq C\frac{h}{p^2}(\frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|\nabla q\|_{0,K}^2) \\ & \leq C(\|q\|_{0,K}^2 + \|\nabla q\|_{0,K}^2). \end{aligned}$$

In particular

$$\begin{aligned} \sum_{e \in F_h^I} \|\frac{1}{\sqrt{\sigma}}q\|_{0,e}^2 & \leq C \sum_{K \in \Pi_h} \|q\|_{0,K}^2 + \|\nabla q\|_{0,K}^2 \\ & \leq C(\|q\|^2 + \|\nabla q\|^2) \\ & \leq C\|u\|^2 \end{aligned}$$

In the same manner, using the imbedding of  $H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$  in  $H^1(\Omega)^3$ ; we can bound  $\sum_{e \in F_h} \|\frac{1}{\sqrt{\sigma}}\phi\|_{0,e}^2$  and obtain

$$\begin{aligned} \|\frac{1}{\sqrt{\sigma}}\phi\|_{0,F_h}^2 & \leq C\|\phi\|_{1,\Omega}^2 \leq C\|\phi\|_{H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)}^2 \leq C(\|\nabla \times \phi\|^2 + \|\nabla \cdot \phi\|^2) \\ & \leq C\|\nabla \times \phi\|^2 \\ & \leq C\|u\|^2. \end{aligned}$$

Finally, we obtain

$$\|u\|^2 \leq C(\|\nabla \cdot u\|^2 + \|\nabla \times u\|^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2)^{\frac{1}{2}} \|u\|. \quad \square$$

### 3 The second inequality

**Lemma 3.1** *Let  $u \in H^1(\Pi_h)^3$  and let  $\sigma = \kappa \frac{p^2}{h}$  with  $p \geq 1$  is an integer. Then, there exists  $C$  independent of  $h$  and  $p$  such that*

$$\|u\|^2 \leq C(\|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e \in F_h} \|\sqrt{\sigma}[u]_N\|_{0,e}^2).$$

**Proof:** The proof is similar to the proof in the previous section. But here we use the following orthogonal decomposition formula if  $\Omega$  is simply-connected (see also [2], [4])

$$L^2(\Omega)^3 = H(\nabla \times 0, \Omega) \oplus H_0(\nabla \cdot 0, \Omega).$$

Then, for  $u \in L^2(\Omega)^3$  we write

$$u = u_1 + u_2$$

with  $u_1 \in H(\nabla \times 0, \Omega)$  and  $u_2 \in H_0(\nabla \cdot 0, \Omega)$ . Since  $\nabla \times u_1 = 0$ , we write  $u_1 = \nabla q$  with  $q \in H^1(\Omega)$  and since  $u_2 \in H_0(\nabla \cdot 0, \Omega)$ , we write  $u_2 = \nabla \times \varphi$  with  $\varphi \in H_0(\nabla \times, \Omega) \cap H(\nabla \cdot 0, \Omega)$  (see [2], [4]).

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