

**Electromagnetic imaging: Representation formulas
for the field perturbations caused by low volume
fraction inhomogeneities**

Michael S. Vogelius
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$$(A(x)\xi) \cdot \xi \geq c_0 |\xi|^2, \quad \int_{\partial\Omega} \phi = 0, \quad \nu \text{ is outward normal} .$$

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ANSWER 1: – Yes, if A is **isotropic**, i.e., if $A(x) = \gamma(x)I$.

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ANSWER 2: – Only very partially (up to a “pullback” by a diffeomorphism) if A is **anisotropic**. Sylvester (1990), Lee-Uhlmann (1987)[“positive”], Kohn-Vogelius, Tartar (1987)[“negative”].

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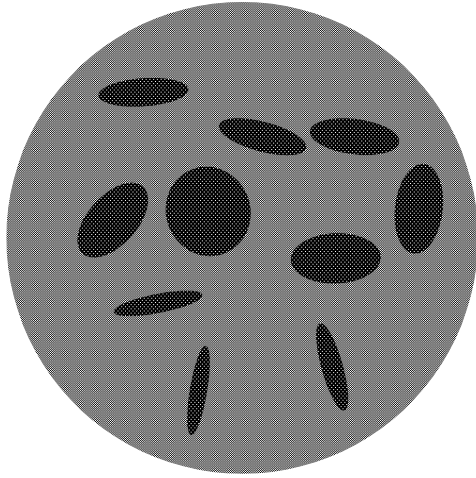
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Collaborators: H. Ammari, E. Beretta, M. Brühl, Y. Capdeboscq, E. Francini, A. Friedman, M. Hanke, D. Hansen, S. Moskow, D. Volkov

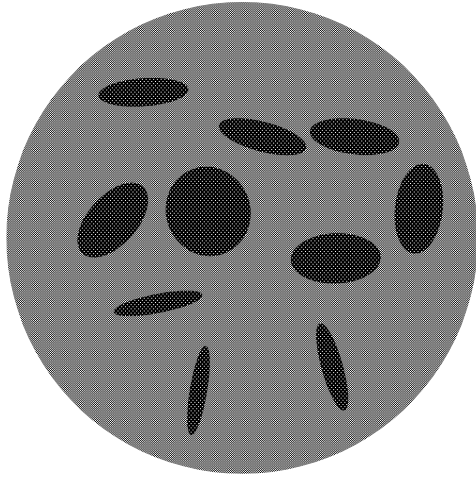


$$\gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & \text{in } \Omega \setminus \omega_\epsilon \\ \gamma_1(x) & \text{in } \omega_\epsilon \end{cases}$$

$$\omega_\epsilon \subset K_0 \subset \Omega$$

$$0 < \gamma_0(x), \gamma_1(x) < \infty$$

$$\begin{cases} \nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \phi & \text{on } \partial\Omega. \end{cases}$$



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GOAL: Find an asymptotic expression for $(u_\epsilon - u_0)|_{\partial\Omega}$ that can be used to determine ω_ϵ (for $|\omega_\epsilon|$ small).

General Representation Formula

After the extraction of a subsequence:

$$\forall y \in \partial\Omega, \quad u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega \quad + o(|\omega_\epsilon|)$$

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$N(x, y)$ is the Neumann function for $\nabla \cdot (\gamma_0 \nabla)$:

$$\nabla_x \cdot (\gamma_0 \nabla_x N(x, y)) = \delta_y \text{ in } \Omega$$

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The probability measure $\mu = \lim_{|\omega_\epsilon| \rightarrow 0} \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon}$ converges weak* in the dual of $C^0(\bar{\Omega})$.

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M is a matrix valued function in $L^2(\Omega, d\mu)$. The values of M are symmetric, positive definite matrices.

Definition of M

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$$\frac{1}{|\omega_{\epsilon}|} 1_{\omega_{\epsilon}} \nabla u_{\epsilon} dx \rightarrow M \nabla u_0 d\mu$$

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But, notice that $v_0^j = x_j + cst$ is not a solution to

$$\begin{cases} \nabla \cdot (\gamma_0 \nabla v_0) = 0 & \text{in } \Omega \\ \gamma_0 \frac{\partial v_0}{\partial \nu} = \psi & \text{on } \partial\Omega. \end{cases}$$

unless γ_0 is a constant!

Definition of M

Instead, construct

$$\left\{ \begin{array}{ll} \nabla \cdot \left(\gamma_\epsilon \nabla v_\epsilon^j \right) = \nabla \cdot \left(\gamma_0 \nabla v_0^j \right) & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial v_\epsilon^j}{\partial \nu} = \gamma_0 \frac{\partial v_0^j}{\partial \nu} & \text{on } \partial\Omega, \\ v_0^j = x_j + \text{cst}, & \end{array} \right.$$

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and define

$$\frac{1}{|\omega_\epsilon|} \mathbf{1}_{\omega_\epsilon} \frac{\partial v_\epsilon^j}{\partial x_k} \xrightarrow{\text{def}} M_{jk} d\mu.$$

To prove $\frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \rightarrow M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$

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 \parallel & \parallel \\
 \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx &= \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx \\
 \downarrow & \\
 \int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu &
 \end{aligned}$$

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$$\begin{array}{ccc}
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+compensated compactness (judicious integration by parts)

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+ cut-offs.

Bounds and variational characterization of M

$$\begin{aligned} \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j d\mu &= \frac{1}{|\omega_\epsilon|} \min_{w \in H^1(\Omega)} \int_{\Omega} \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 dx \\ &\quad + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 dx + o(1) . \end{aligned}$$

Bounds and variational characterization of M

$$\int_{\Omega} (\gamma_1 - \gamma_0) M_{ij} \xi_i \xi_j \phi d\mu = \frac{1}{|\omega_{\epsilon}|} \min_{w \in H^1(\Omega)} \int_{\Omega} \gamma_{\epsilon} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon}} \xi \right|^2 \phi dx \\ + \frac{1}{|\omega_{\epsilon}|} \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx + o(1) .$$

for all uniformly positive, smooth functions ϕ on Ω .

Bounds and variational characterization of M

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This yields

$$\min \left(1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 \leq M(x) \xi \cdot \xi \leq \max \left(1, \frac{\gamma_0(x)}{\gamma_1(x)} \right) |\xi|^2 ,$$

for $\xi \in \mathbb{R}^n$, μ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$.

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Bounds and variational characterization of M

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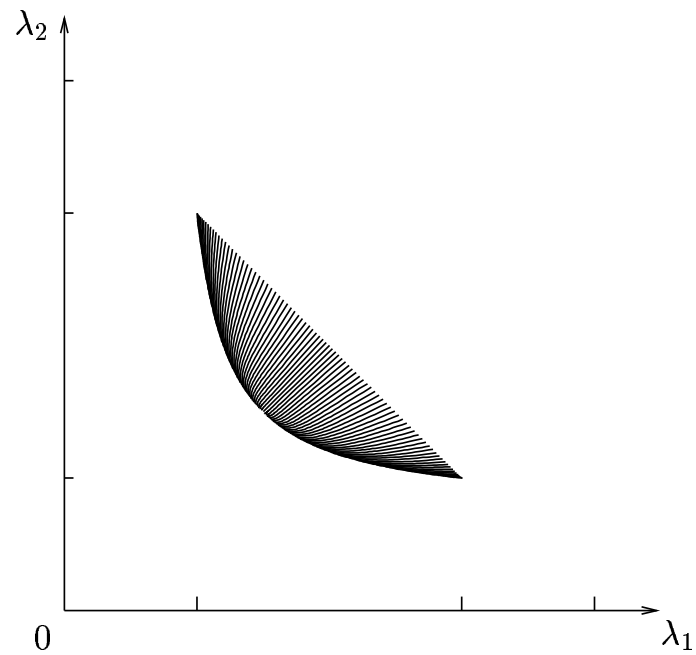
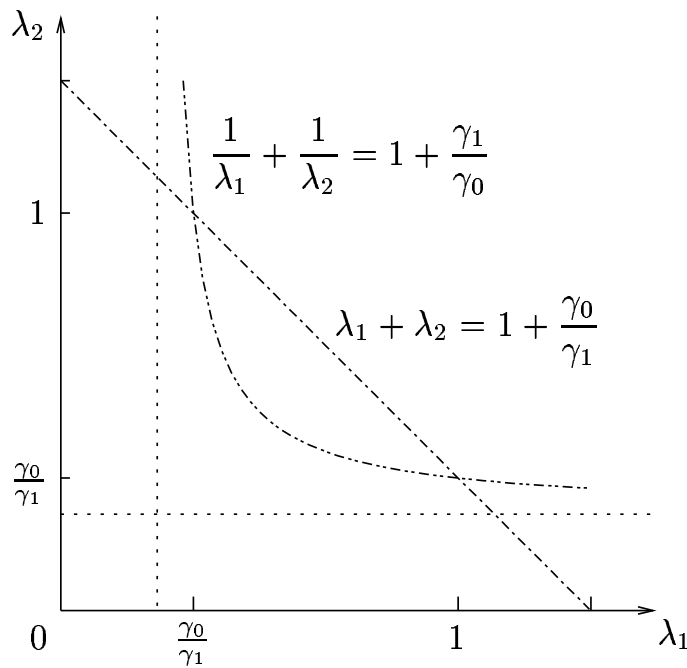
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Bounds

M satisfies

$$\min \left(1, \frac{\gamma_0}{\gamma_1} \right) I_n \leq M \leq \max \left(1, \frac{\gamma_0}{\gamma_1} \right) I_n,$$

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Three of these bounds are attained for a single “sheet-like” inclusion. In that case the polarization eigenvalues “parallel” to the sheet are 1, and the eigenvalue across the sheet is $\frac{\gamma_0}{\gamma_1}$. The fourth bound is attained for a single inclusion in the shape of a ball.

Applications

$$\forall y \in \partial\Omega, u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)$$

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May be used :

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).
2. To estimate the volume of inhomogeneities of moderate size (Capdeboscq, MV)

Detecting locations

Suppose $\omega_\epsilon = \cup_{j=1}^p (z_j + \epsilon B_j)$ (the inhomogeneities “shrink” to points z_j).

$$\begin{aligned} D(\phi)(\cdot) &= |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, \cdot) d\mu(x) \\ &= |\omega_\epsilon| \sum_{j=1}^p (\gamma_1 - \gamma_0) \alpha_j M^j \nabla u_0(z_j) \cdot \nabla_x N(z_j, \cdot) \end{aligned}$$

Is linear in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np).

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Is linear in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np). In fact,

$$\mathcal{R}(D) = \text{span}\{e_k \cdot \nabla_x N(z_j, \cdot)|_{\partial B} : k = 1, \dots, n, j = 1, \dots, p\}.$$

Probe with $g_{z,d} = d \cdot \nabla_x N(z, \cdot)|_{\partial B}$. Then $g_{z,d} \in \mathcal{R}(D)$ iff

$z \in \{z_j : j = 1, \dots, p\}$. Note also that $\mathcal{R}(D)$ is well approximated by

$\mathcal{R}(\Lambda_\epsilon - \Lambda_0)$ (the measured Neumann-Dirichlet “difference” map).

Detecting locations

Method:

Detecting locations

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1. Compute the SVD decomposition of $\Lambda_\epsilon - \Lambda_0$, and the projector onto the space spanned by the first m eigenvectors, P_m .

Detecting locations

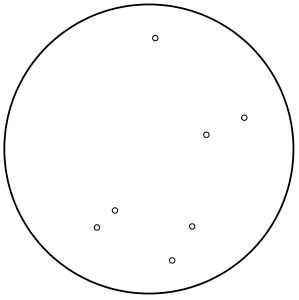
Method:

1. Compute the SVD decomposition of $\Lambda_\epsilon - \Lambda_0$, and the projector onto the space spanned by the first m eigenvectors, P_m .
2. For a test point z , compute

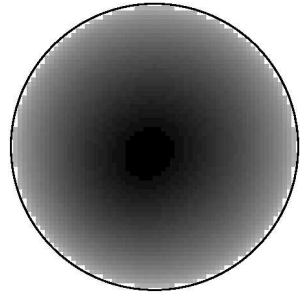
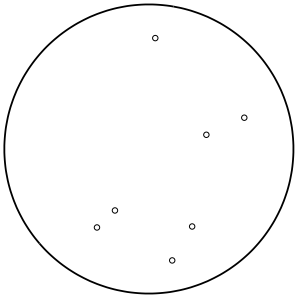
$$\cot \theta_m(z) = \frac{\|P_m g_{z,d}\|}{\|(I - P_m)g_{z,d}\|}.$$

For $m = pn$, $z \in \{z_j : j = 1, \dots, p\} \Leftrightarrow \cot \theta_m(z) = \infty$.

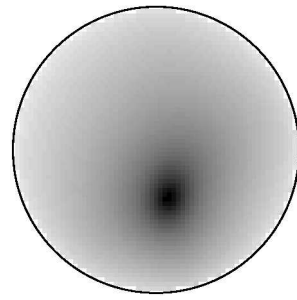
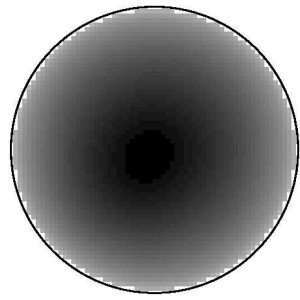
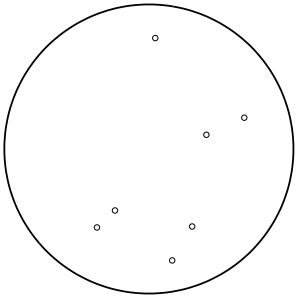
Detecting locations



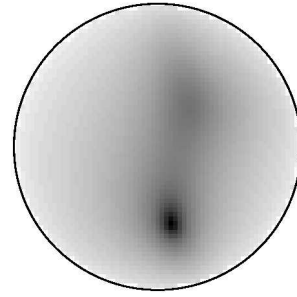
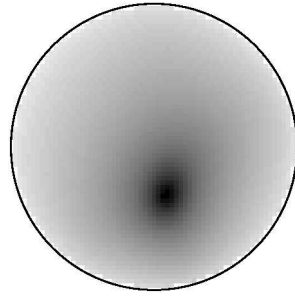
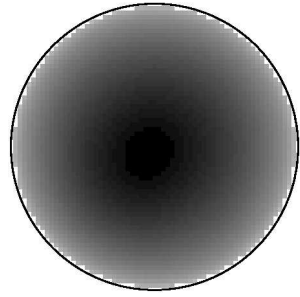
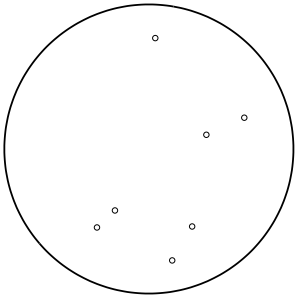
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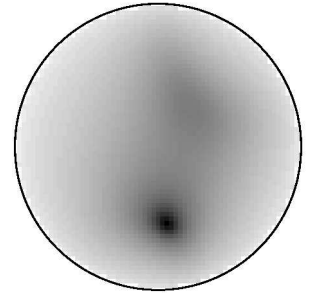
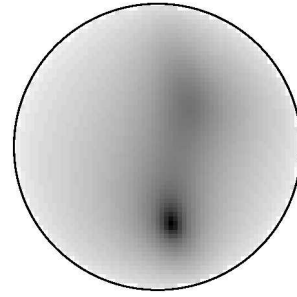
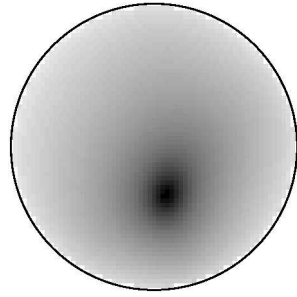
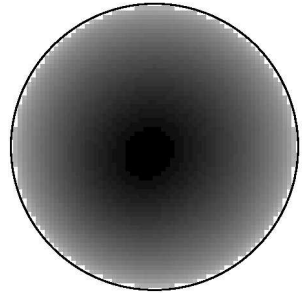
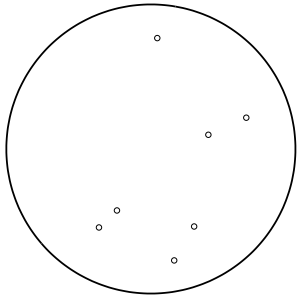
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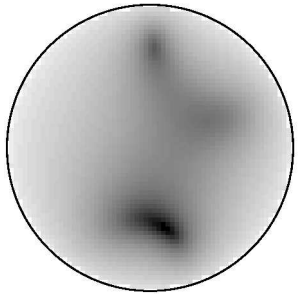
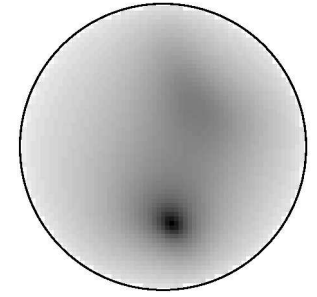
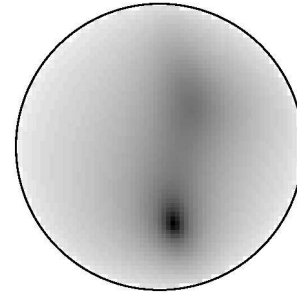
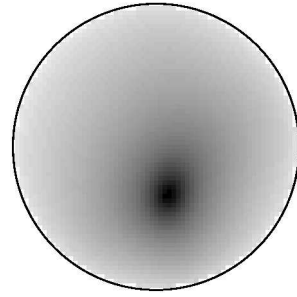
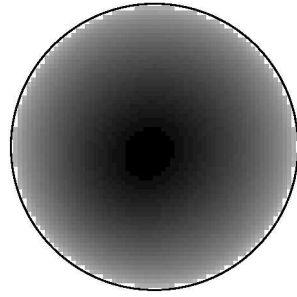
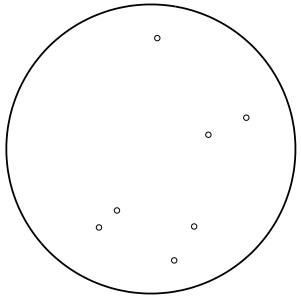
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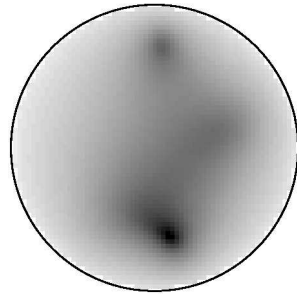
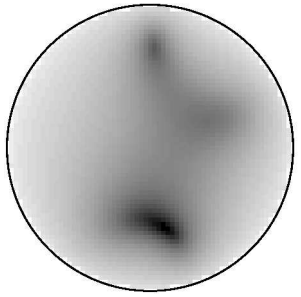
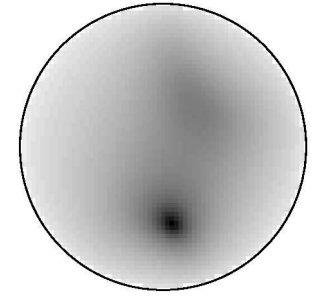
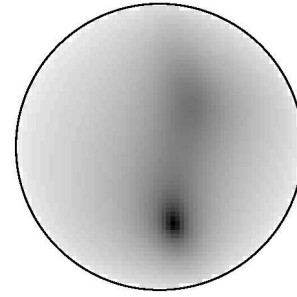
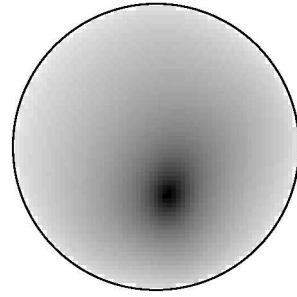
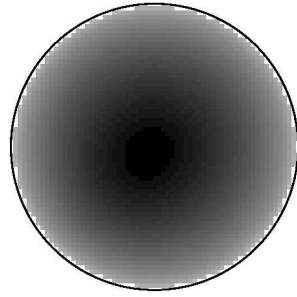
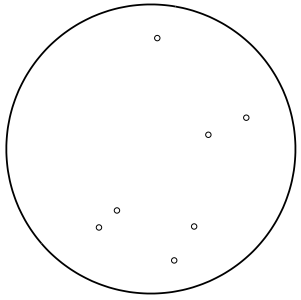
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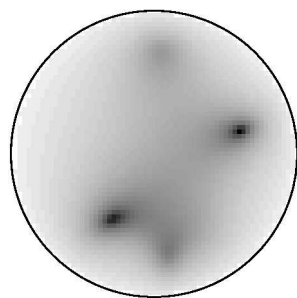
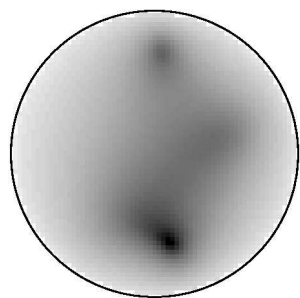
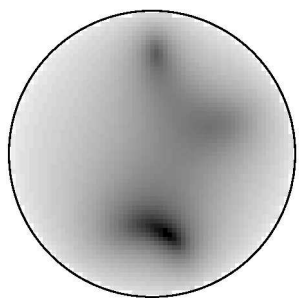
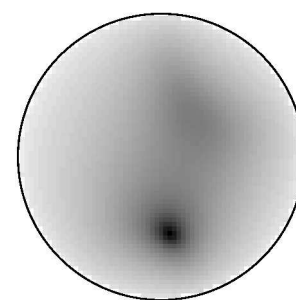
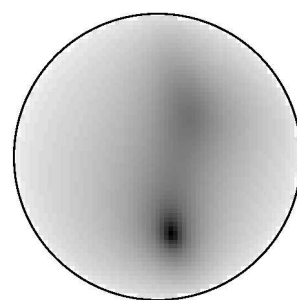
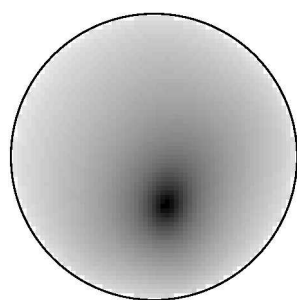
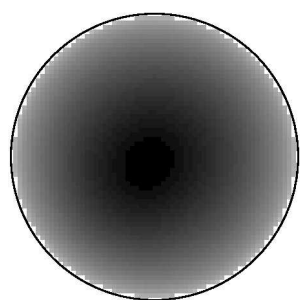
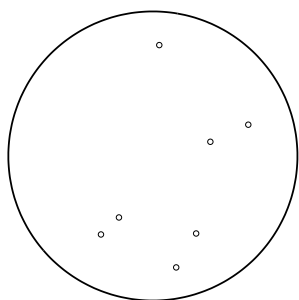
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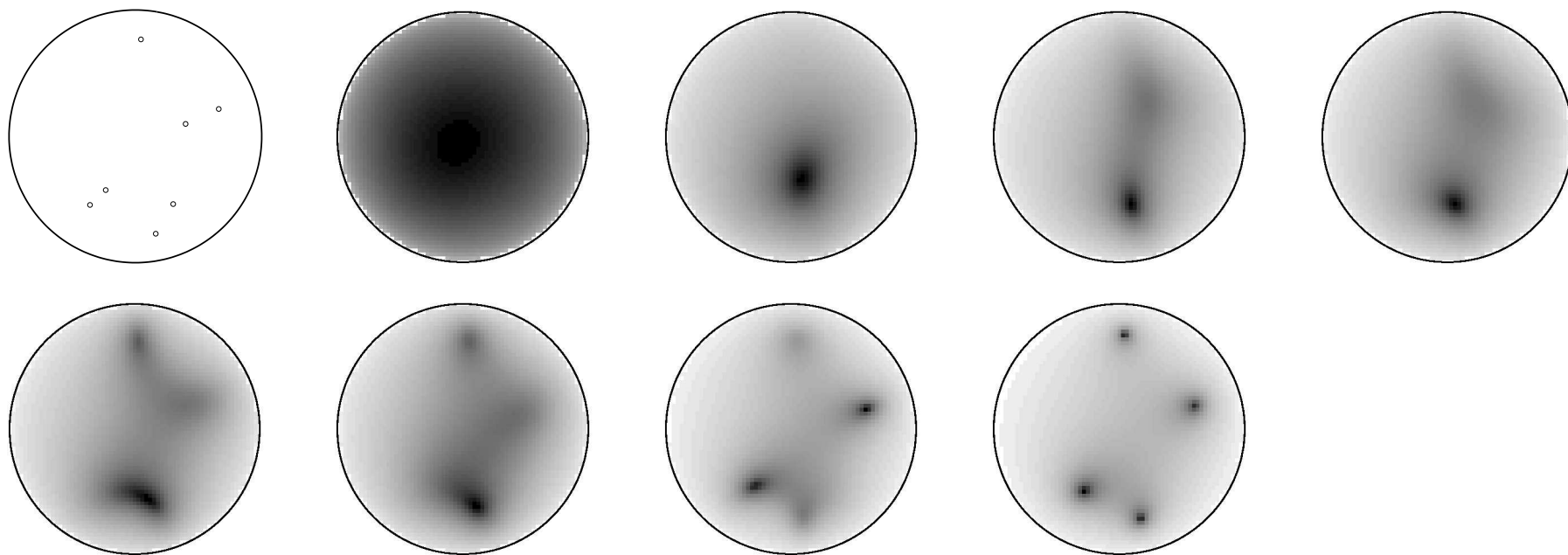
Detecting locations



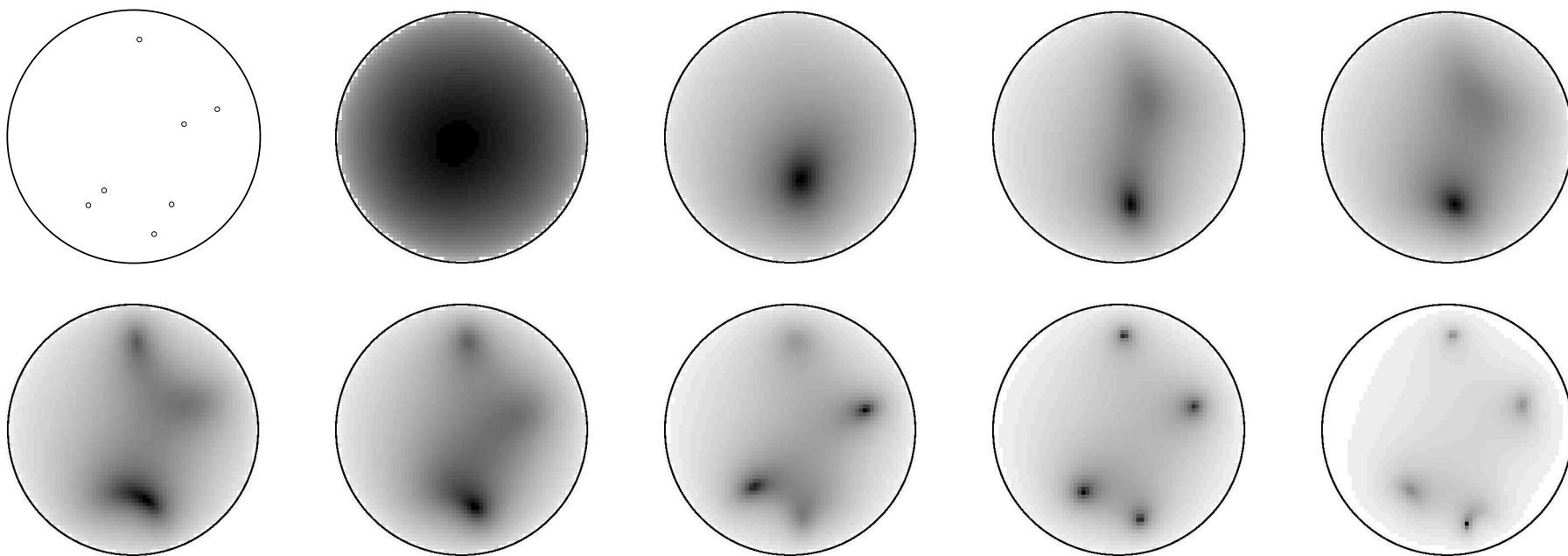
Detecting locations



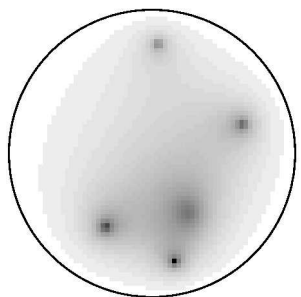
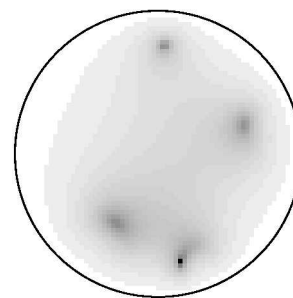
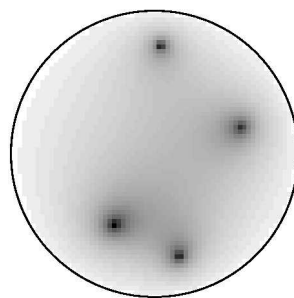
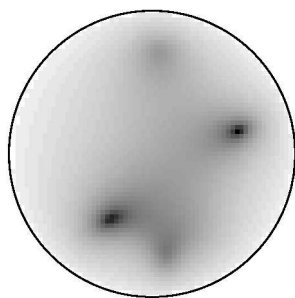
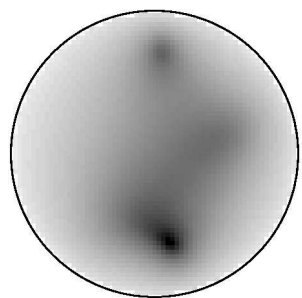
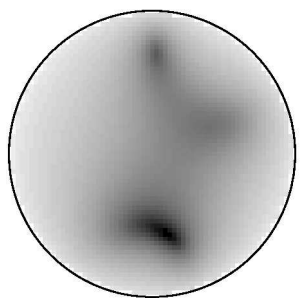
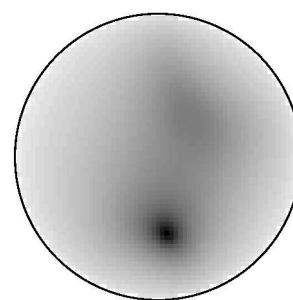
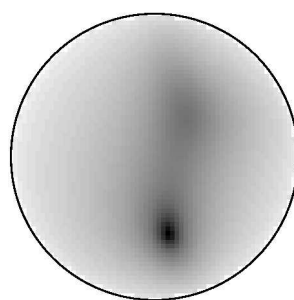
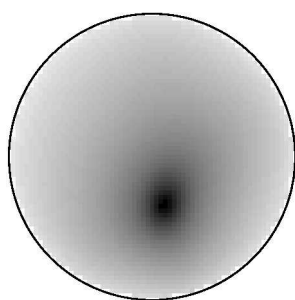
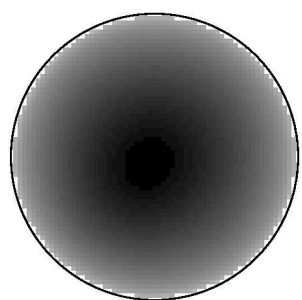
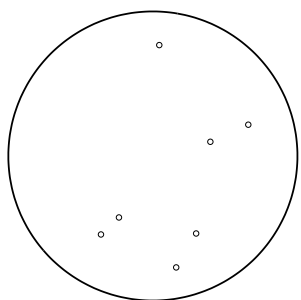
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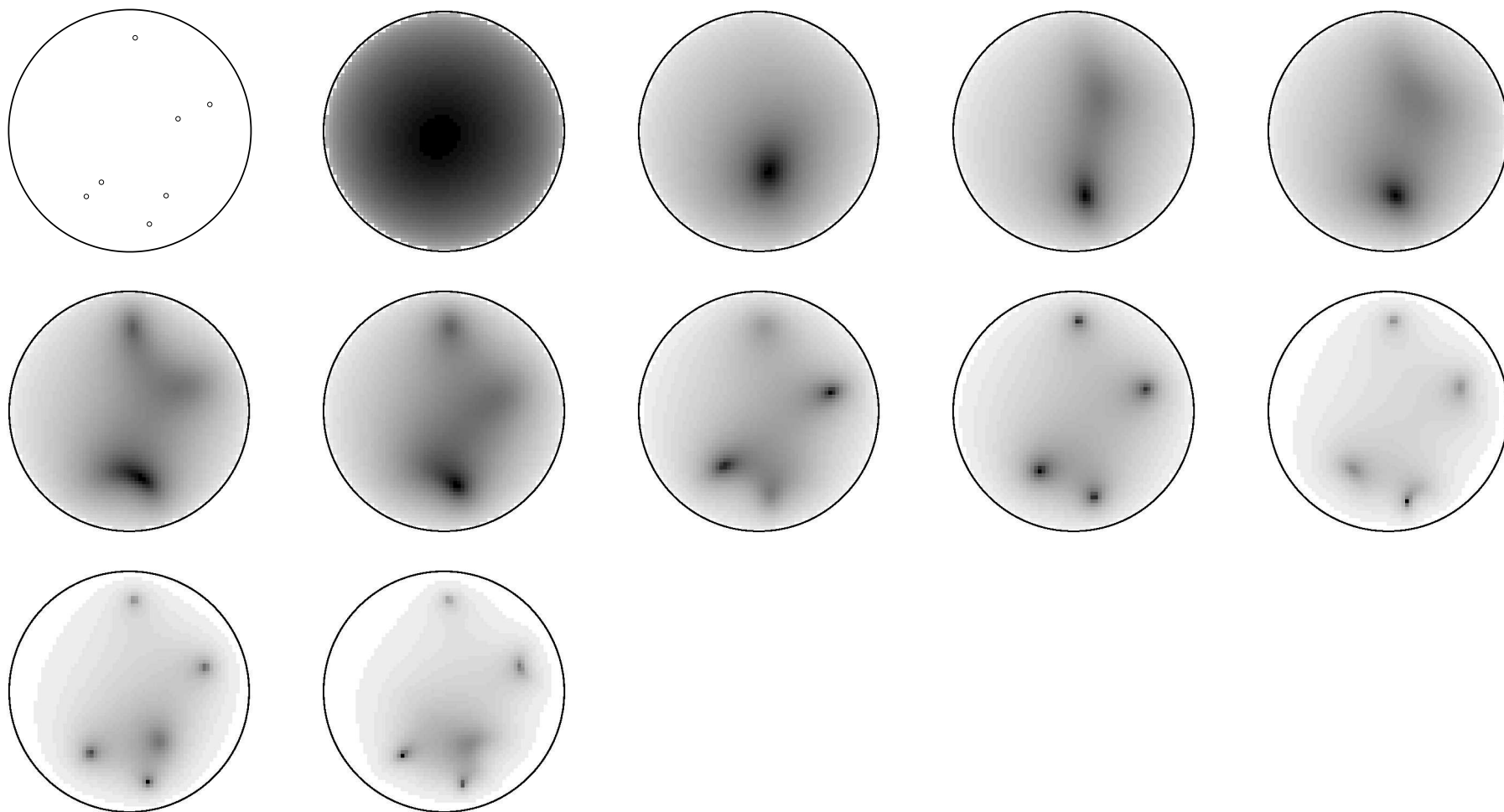
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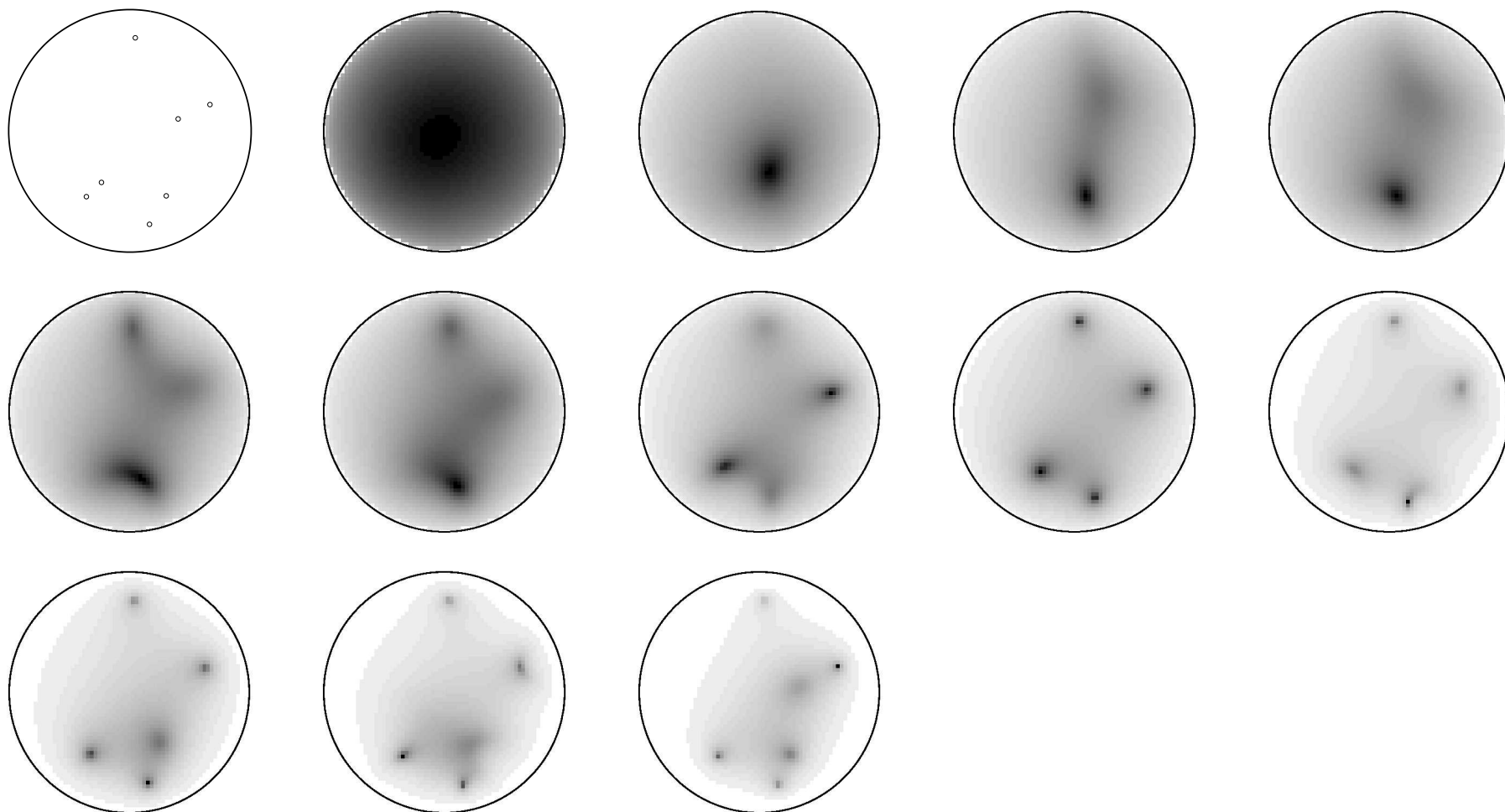
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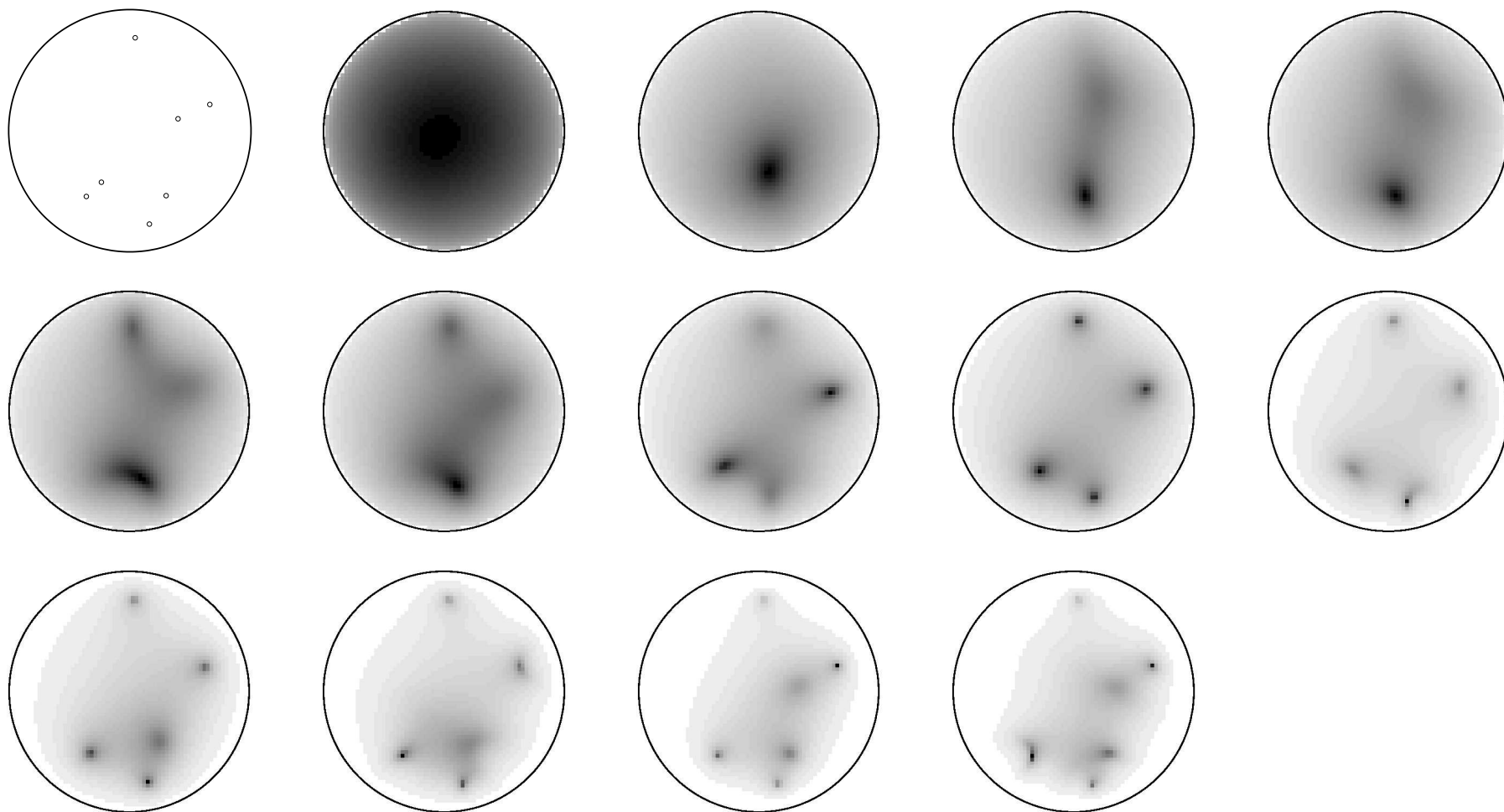
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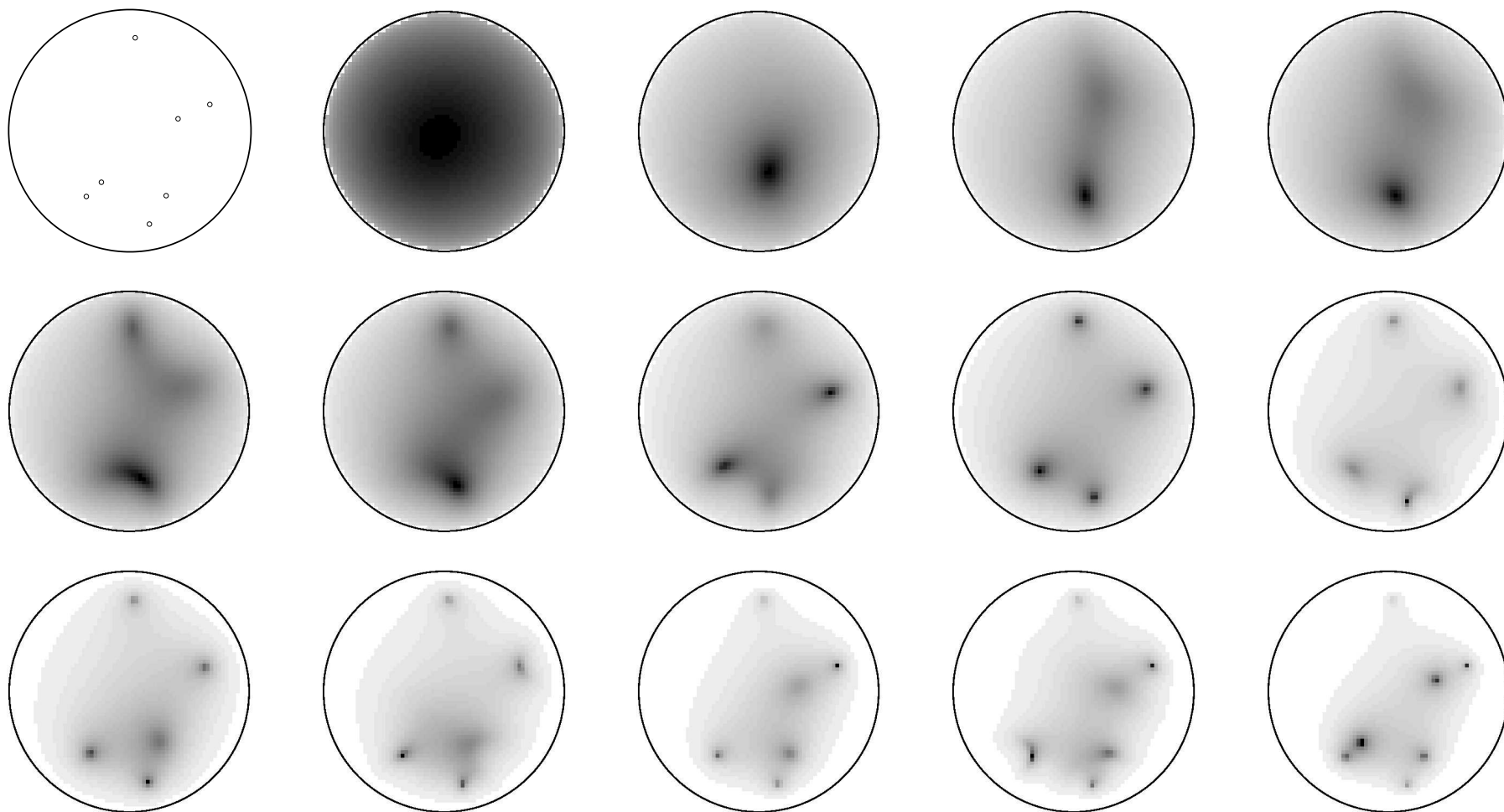
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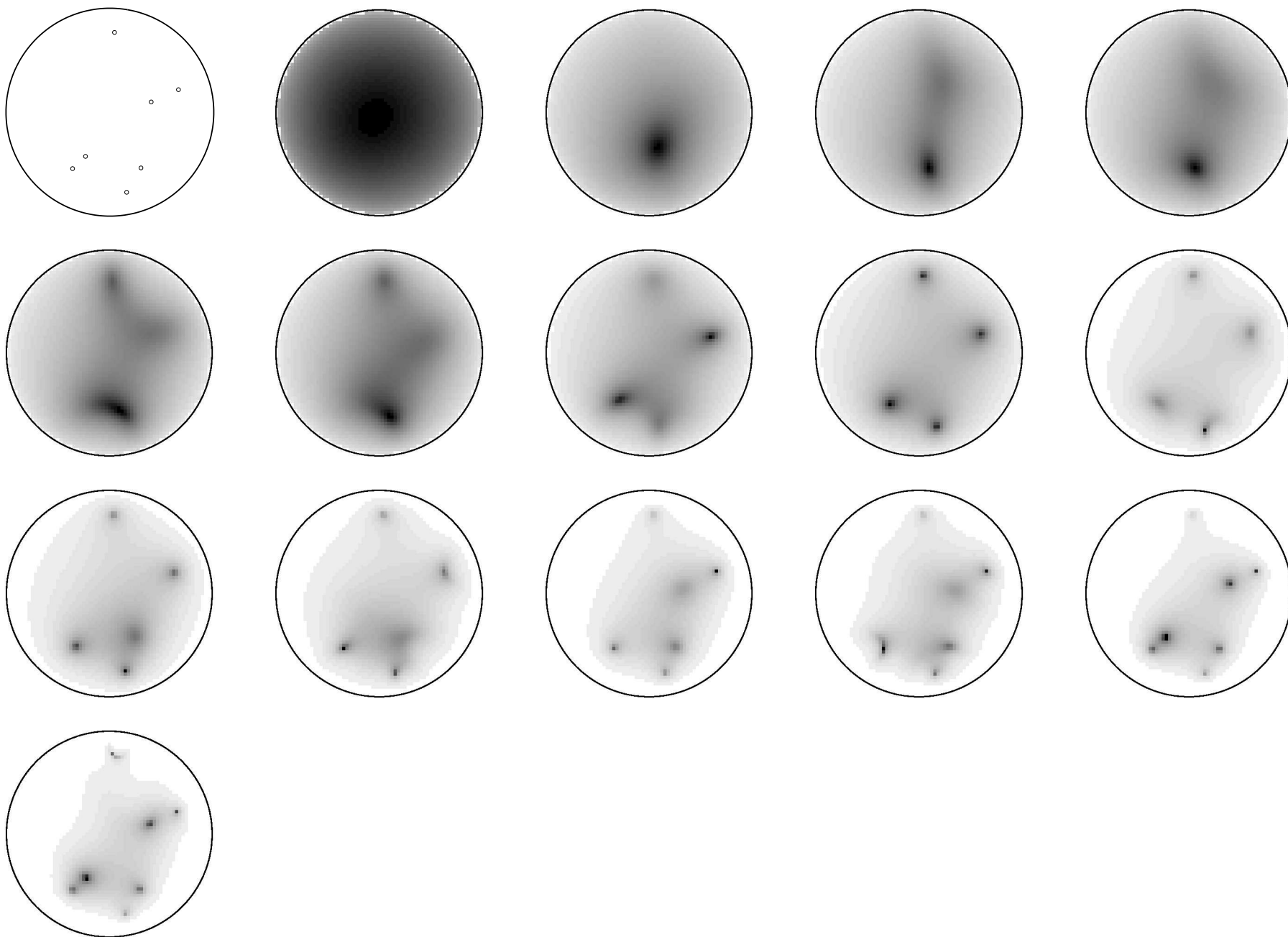
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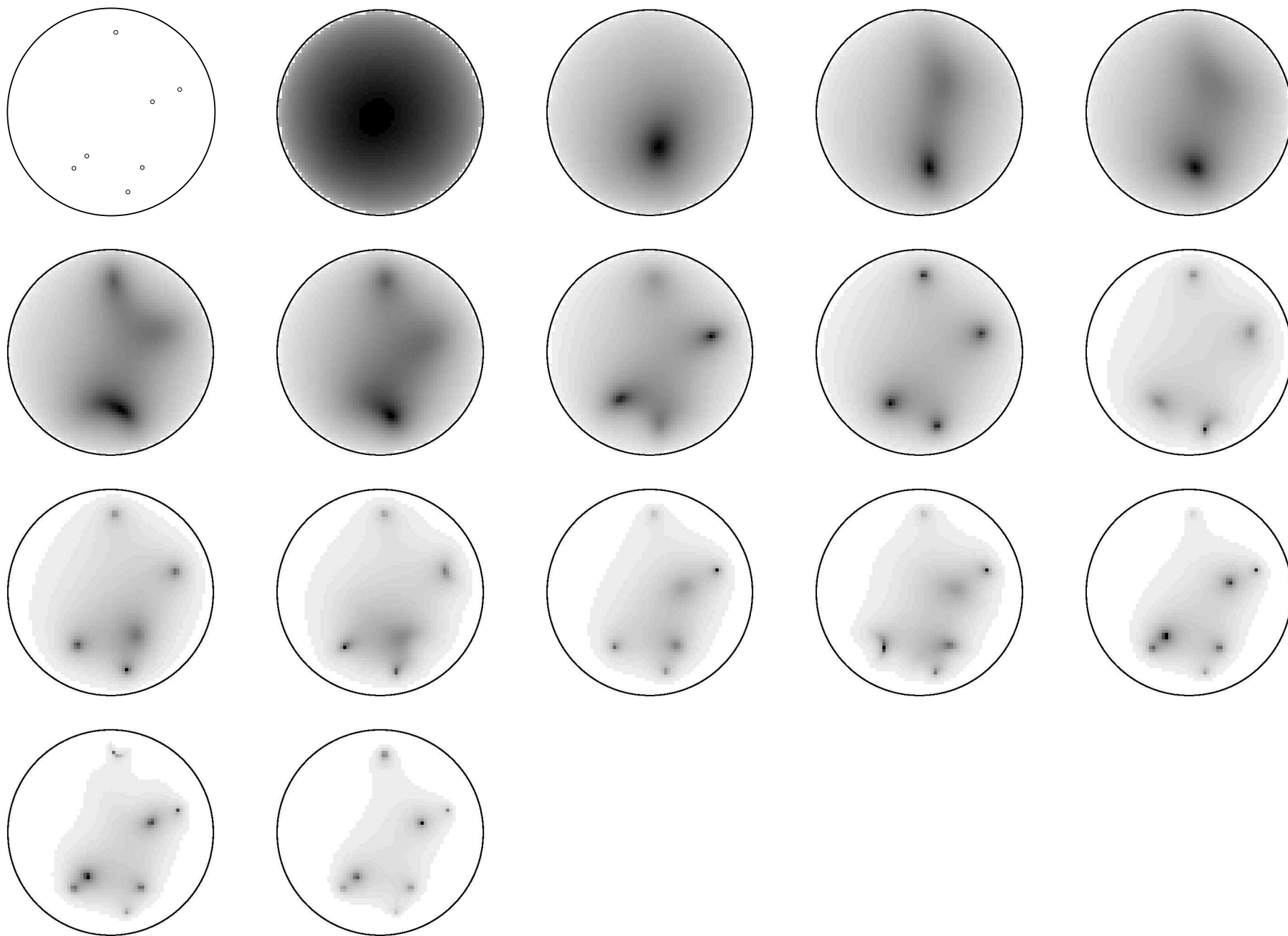
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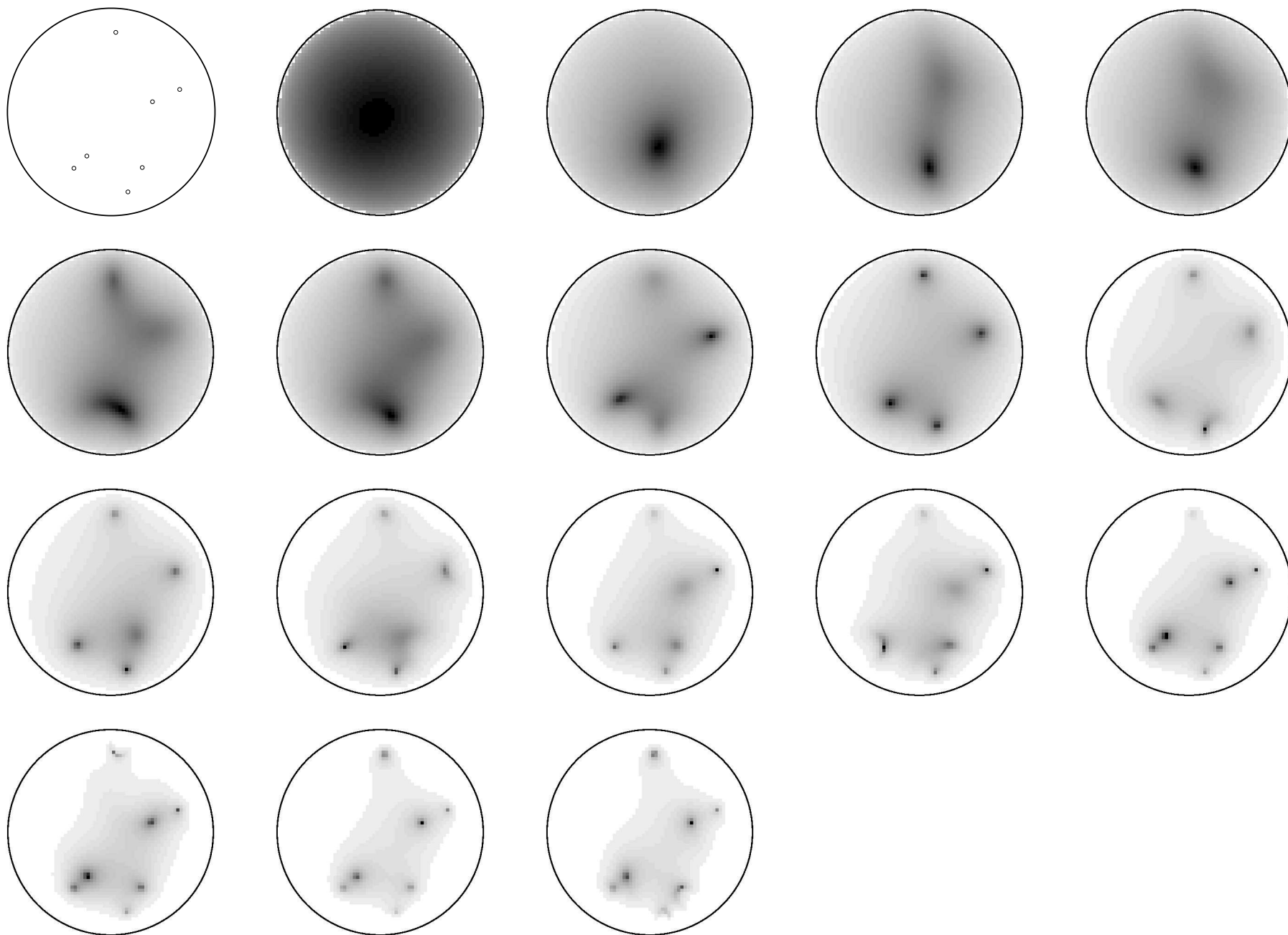
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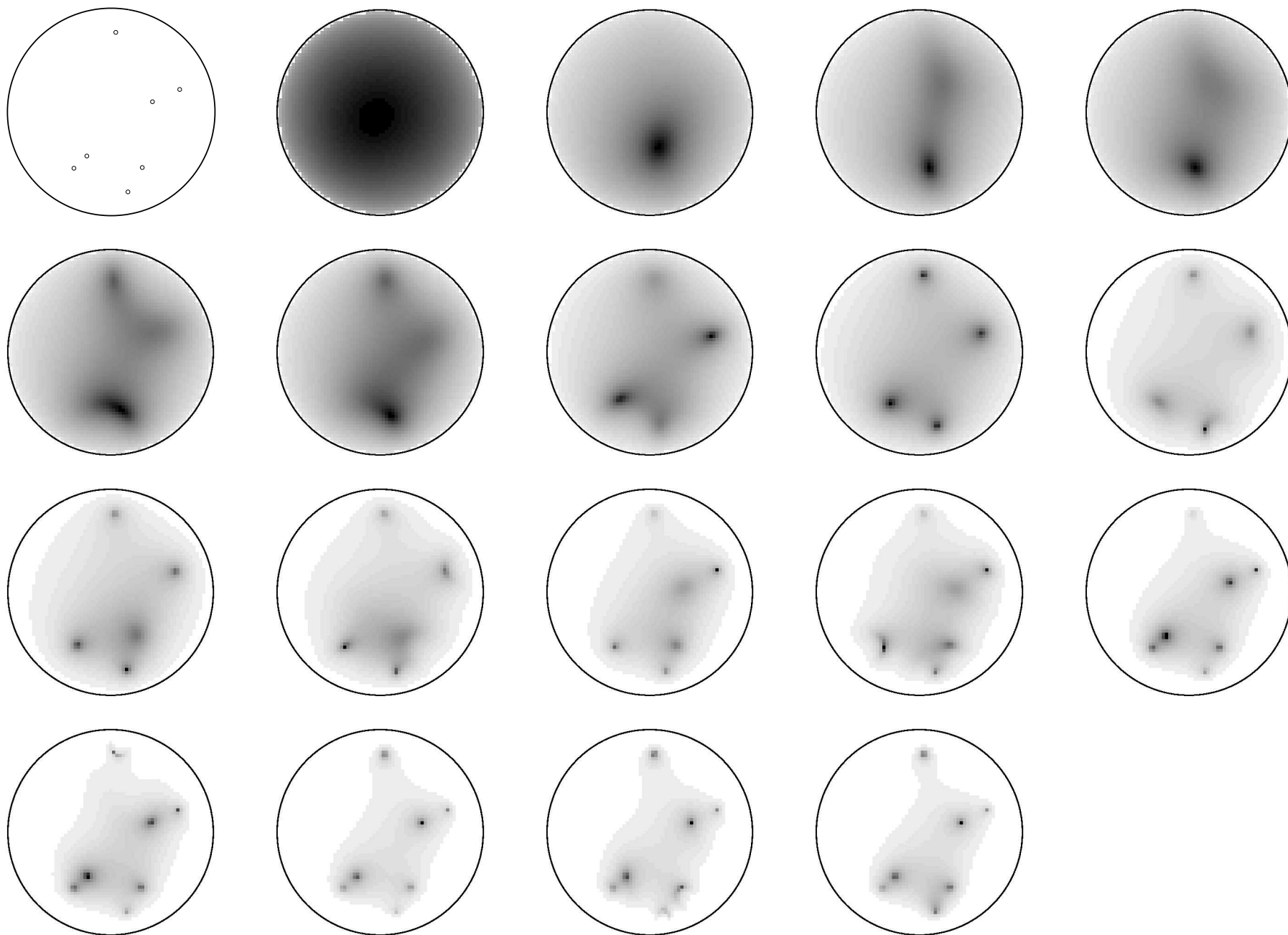
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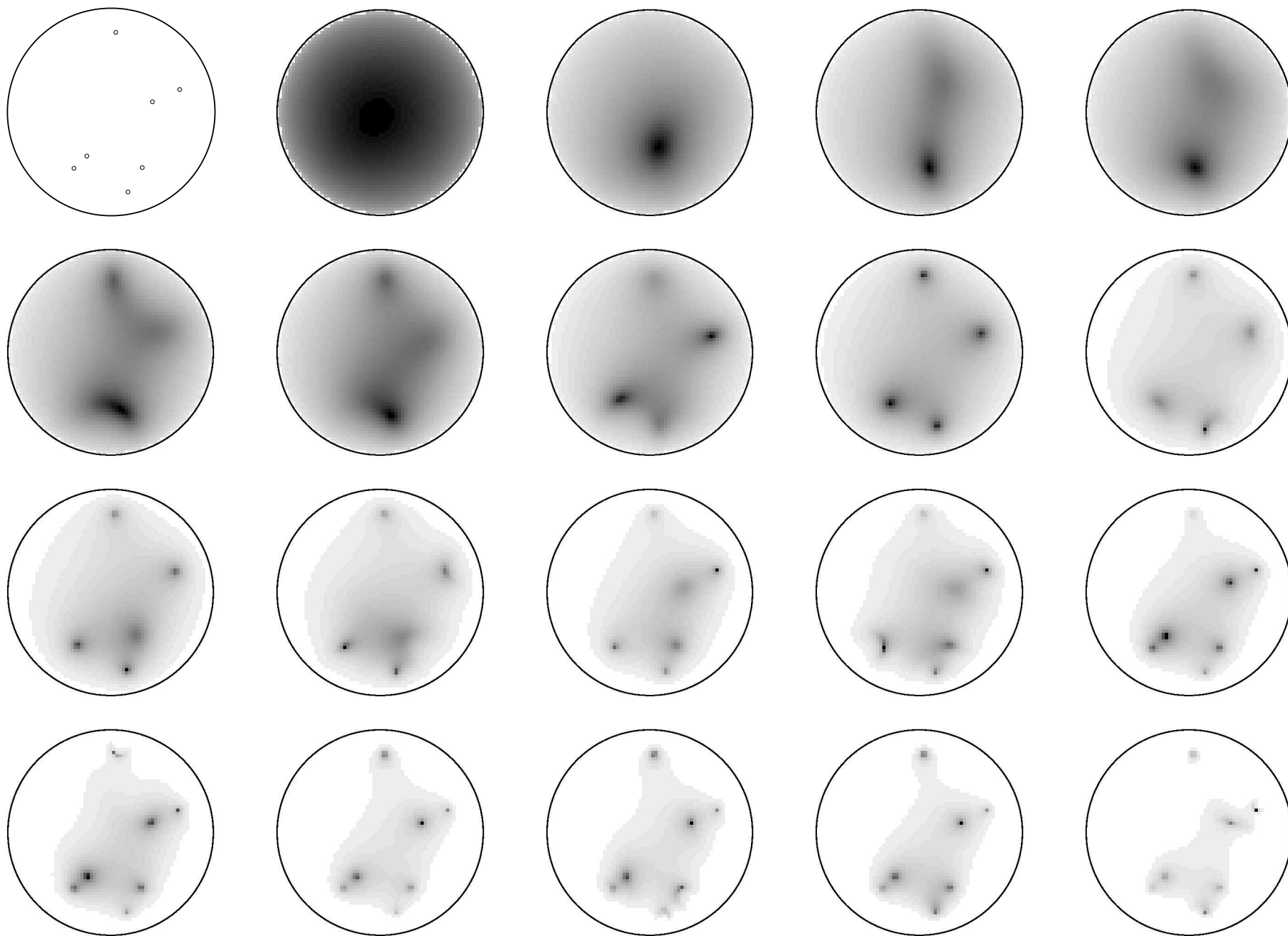
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Volume estimation

Suppose γ_0 and γ_1 are constants.

$$u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x)$$

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$$\left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \int_{\Omega} M_{jj} d\mu, \quad \text{and} \quad \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \text{trace} \left(\int_{\Omega} M d\mu \right).$$

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Using the bounds on M we obtain

$$\min \left(1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq \max \left(1, \frac{\gamma_1}{\gamma_0} \right) \left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right|.$$

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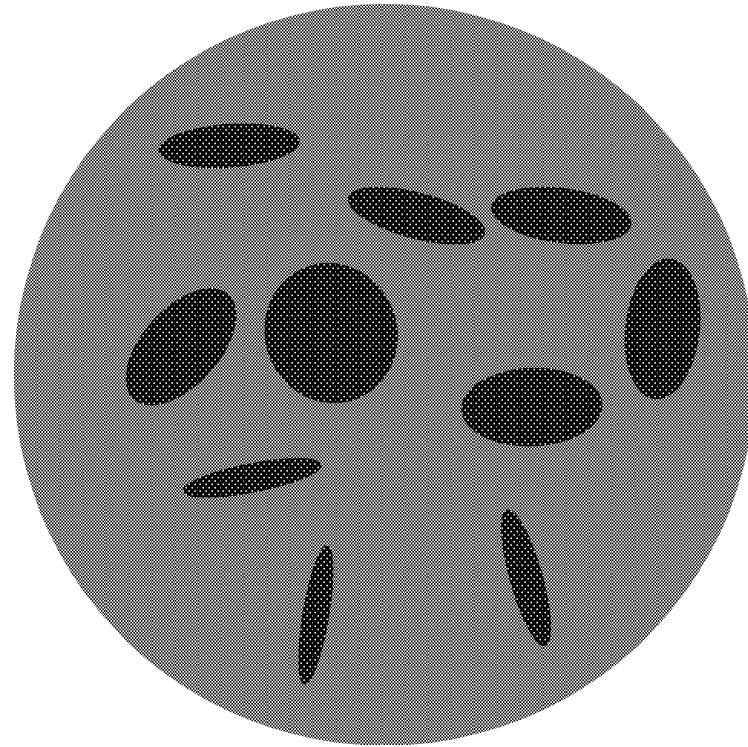
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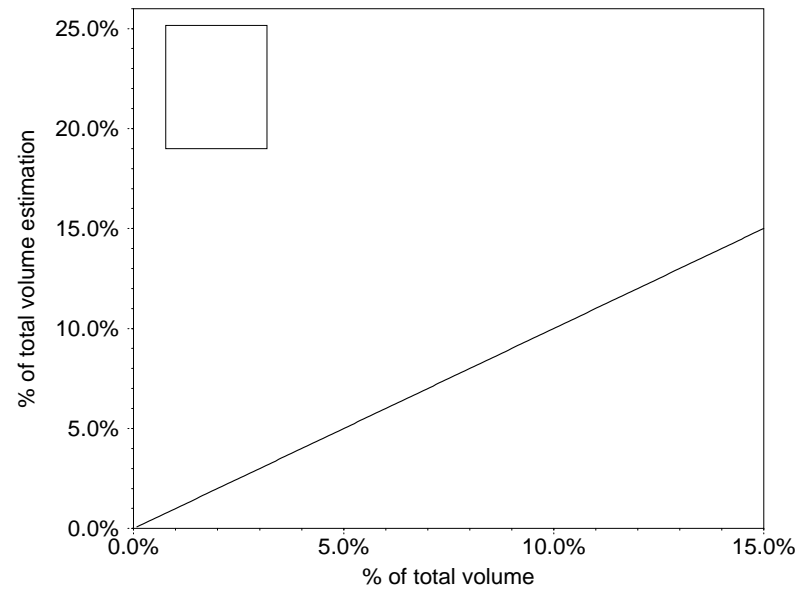
$$\frac{h \left(1, \dots, 1, \frac{\gamma_1}{\gamma_0} \right)}{n} \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right| \leq |\omega_\epsilon| \leq \frac{a \left(1, \dots, 1, \frac{\gamma_1}{\gamma_0} \right)}{n} \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right|$$

n measurement bounds. Only asymptotic bounds.

Random Ellipses

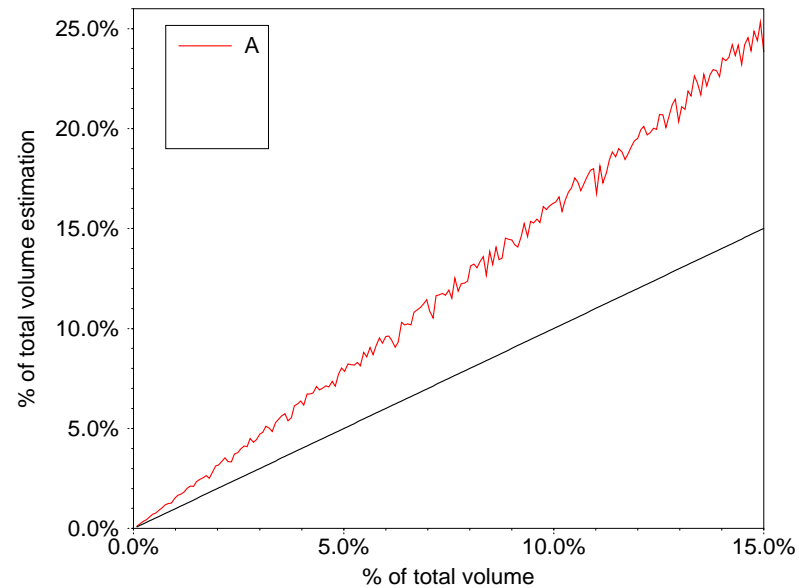


Volume estimation, with $\gamma_0 > \gamma_1$



Proportion of total volume occupied by the inhomogeneities.

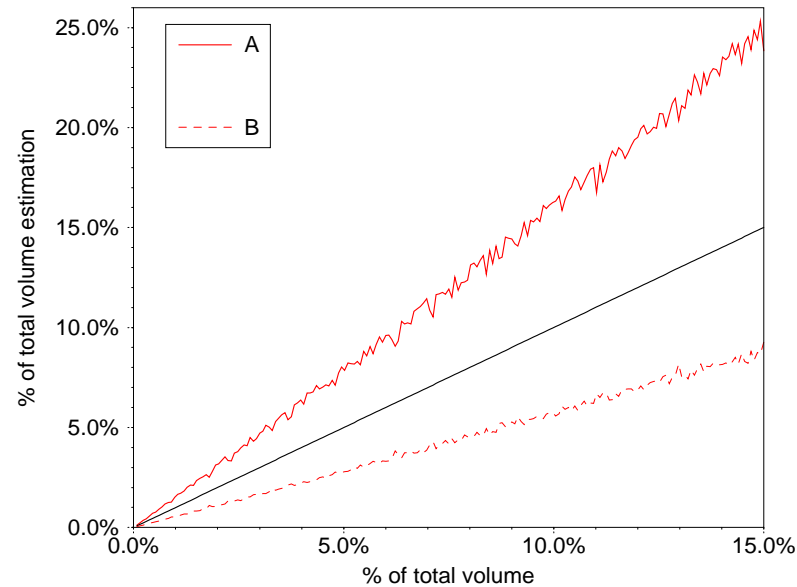
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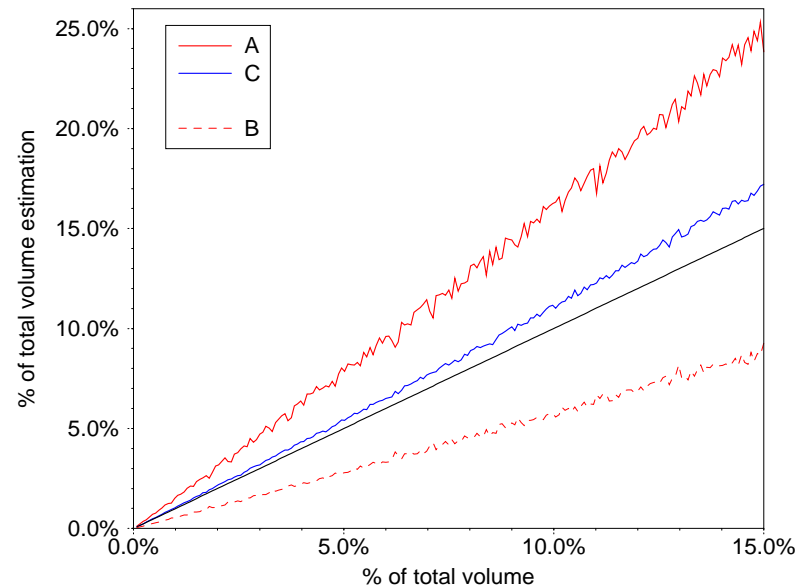


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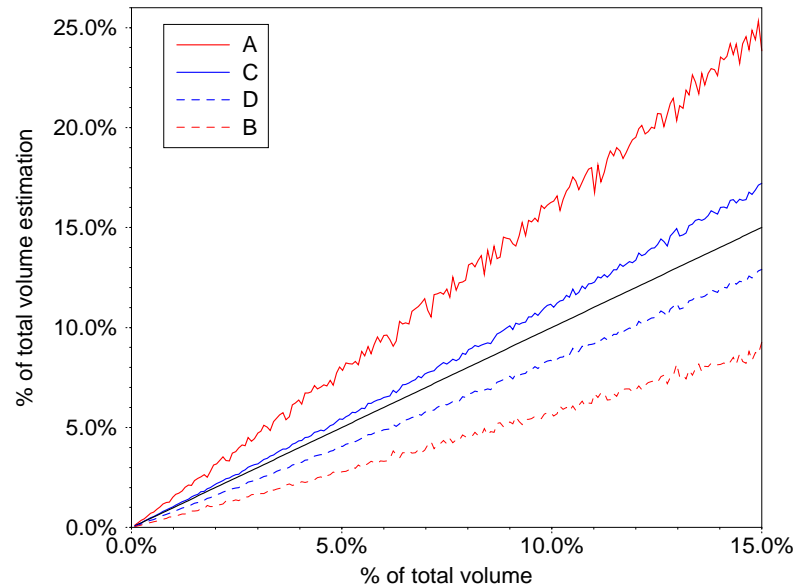
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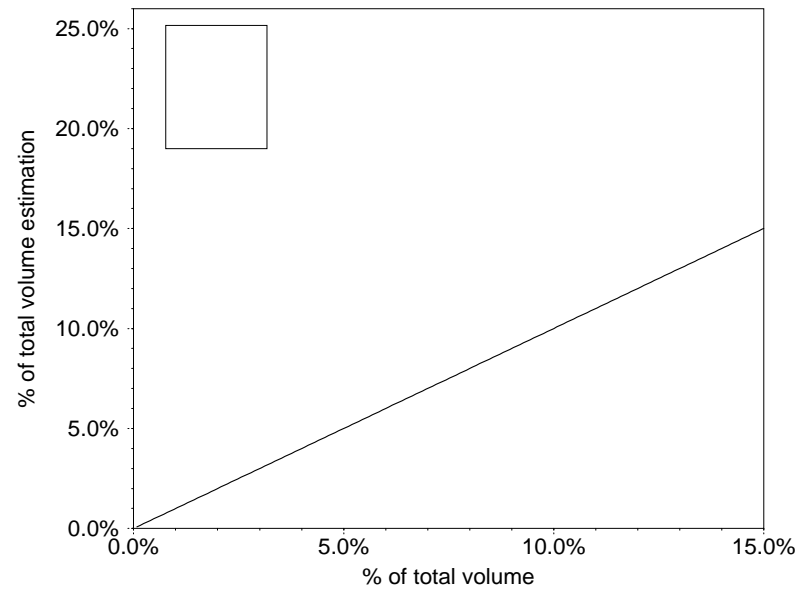
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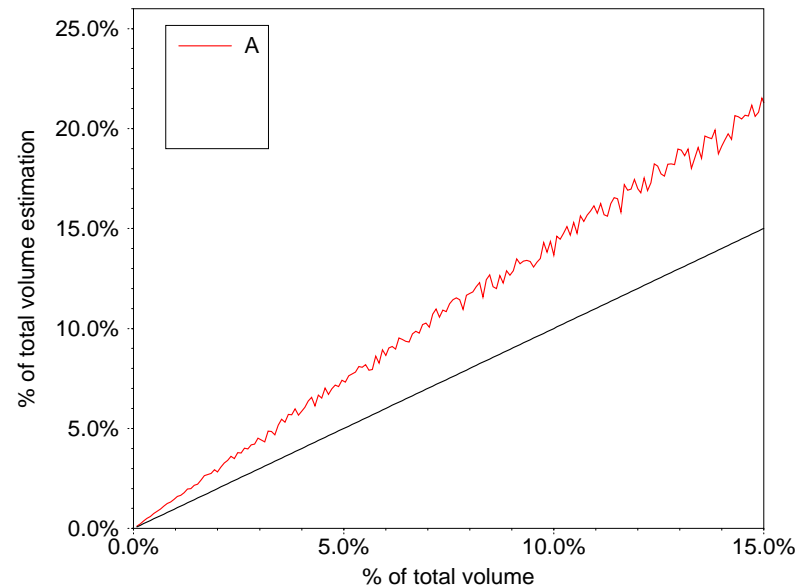
D: the lower estimate obtained from two measurement bounds.

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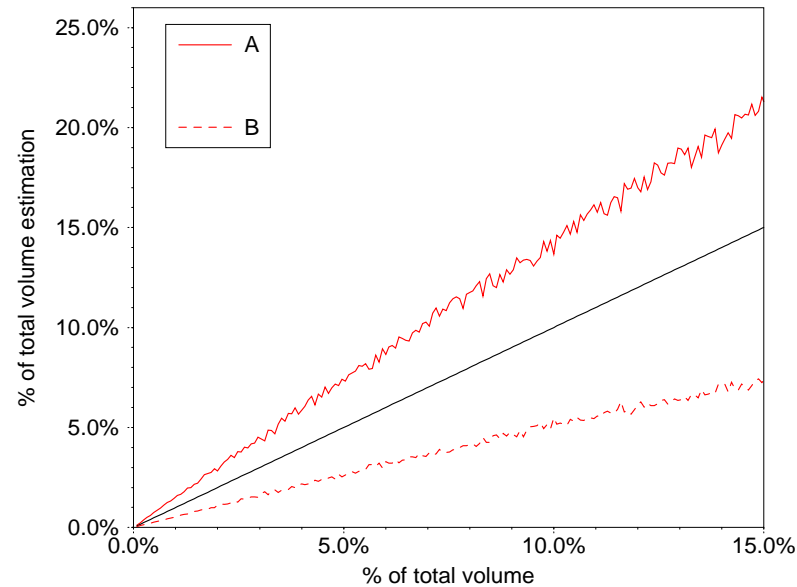
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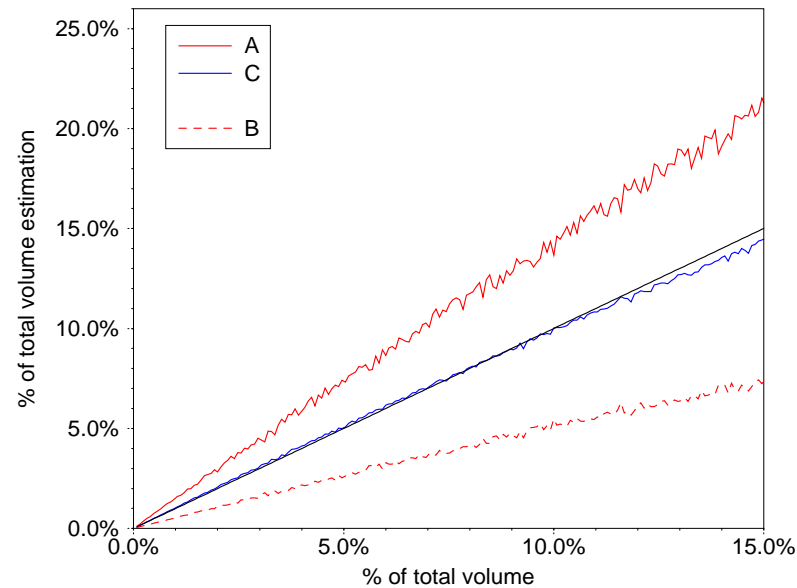


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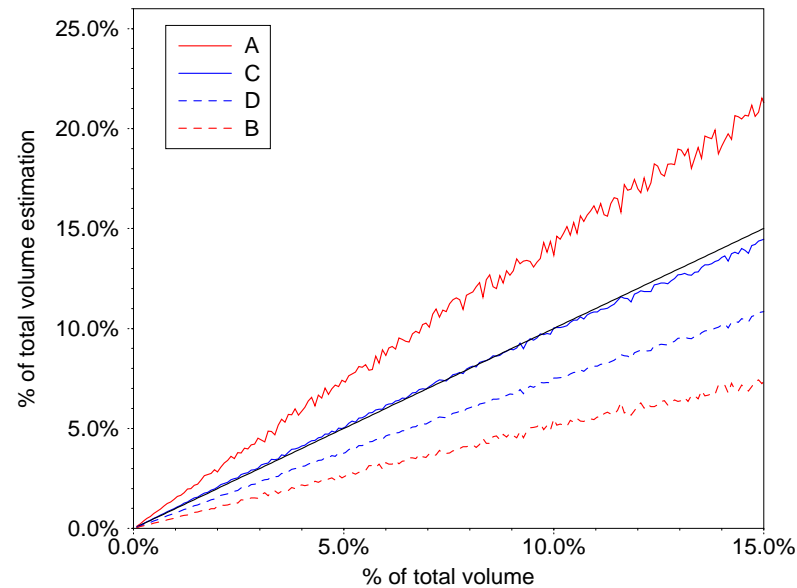
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We now take E_0 to be a plane wave, *i.e.*,

$$\Phi^k(x, y) = -\frac{i}{4} H_0^{(1)}(k \sqrt{\mu_0 q_0} |x - y|) \quad \text{and} \quad E_0(x) = e^{ik \sqrt{\mu_0 q_0} \xi \cdot x}$$

The first two terms are now of order $(\epsilon k)^2 / (1 + \sqrt{k})$ and it is possible to prove that the remainder is of order $o((\epsilon k)^2) / (1 + \sqrt{k})$, as $\epsilon k \rightarrow 0$,

$k \geq k_0 > 0$.

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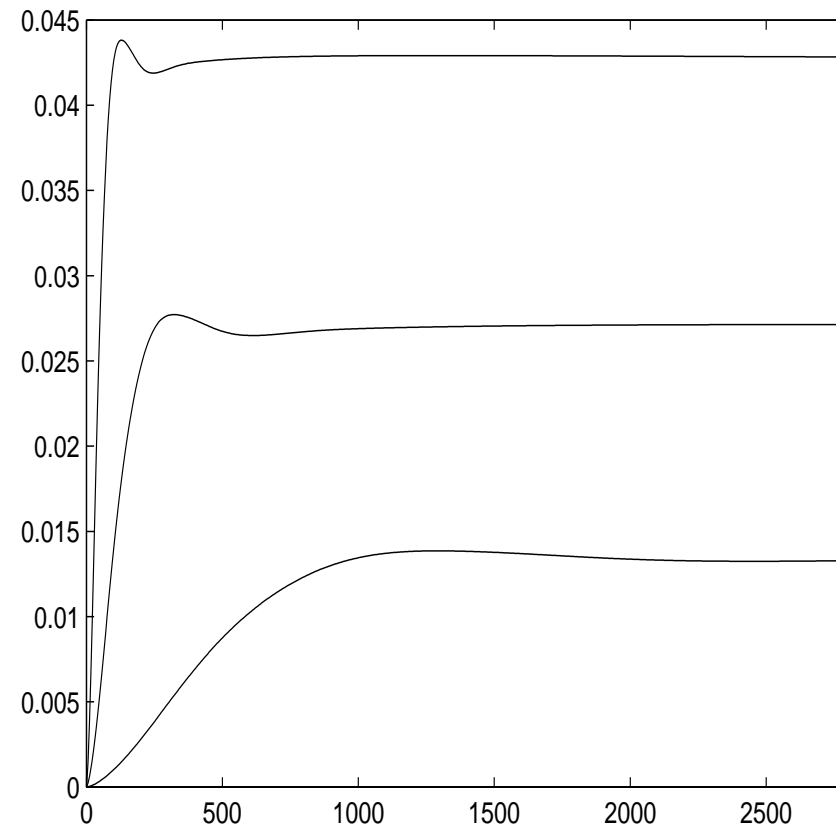
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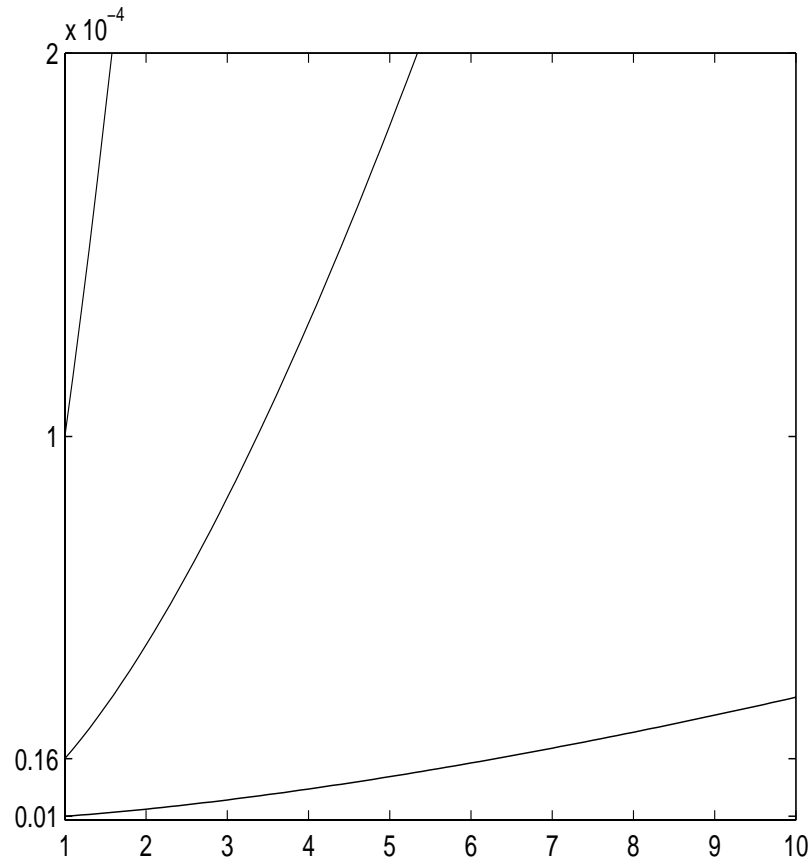
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We now show the L^2 norm of $E_\epsilon(y) - E_0(y)$ on the circle $|y| = 2$ for three circular inhomogeneities of radius $\epsilon = 0.01$, $4 \cdot 0.001$, and 0.001 , respectively.

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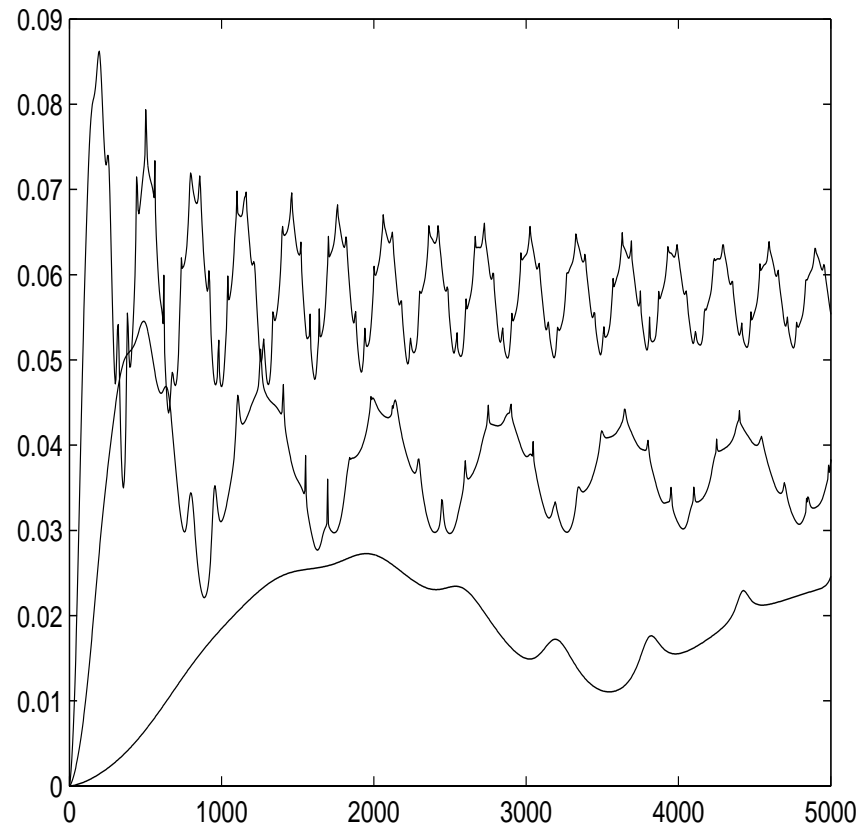


The 2d Helmholtz Equation (Transverse Magnetic Maxwell)



These were for a conducting inhomogeneity ($\mu_1 = 2$, $q_1 = 2 + 2i$ inside the inhomogeneity, $\mu_0 = q_0 = 1$ outside). For a non-conducting inhomogeneity a similar computation gives

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The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

One inhomogeneity of the form ϵD ; change variables $x \rightarrow z = \frac{x}{\epsilon}$,
 $k \rightarrow \lambda = k\epsilon$, $V_\lambda(z) = E_{k,\epsilon}(z\epsilon)$. Then

$$\nabla_z \cdot \left(\frac{1}{\mu} \nabla_z V \right) + \lambda^2 q V = 0 \quad \text{in } \mathbb{R}^2 .$$

Introduce $V_\lambda^{(s)}(z) = V_\lambda(z) - e^{i\lambda\sqrt{\mu_0 q_0} \xi \cdot z}$, then

$$\begin{aligned} E_{k,\epsilon}(y) - E_{k,0}(y) &= V_\lambda^{(s)}\left(\frac{y}{\epsilon}\right) \\ &= \int_{\partial D} \frac{\partial}{\partial n_x} \Phi^\lambda\left(x, \frac{y}{\epsilon}\right) V_\lambda^{(s)}(x) d\sigma_x \\ &\quad - \int_{\partial D} \Phi^\lambda\left(x, \frac{y}{\epsilon}\right) \frac{\partial}{\partial n} V_\lambda^{(s)}(x) d\sigma_x \end{aligned}$$

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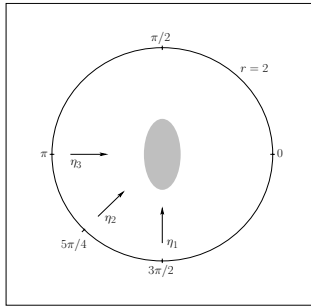
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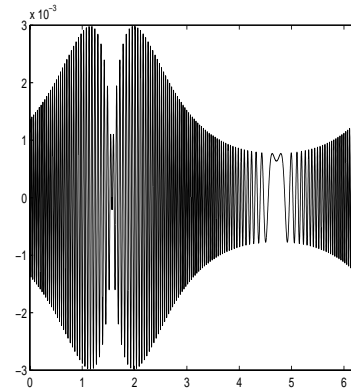
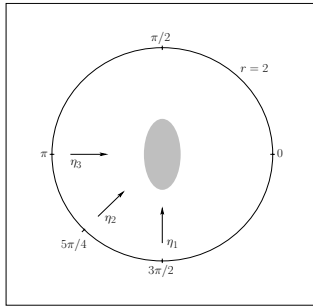
- (1) $\lambda \rightarrow 0$ gives the low frequency formulas from before,
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- (3) $\lambda \rightarrow \infty$ may, e.g., be treated by a combination with appropriate “geometric optics” and stationary phase.

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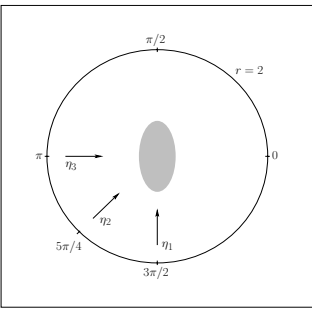
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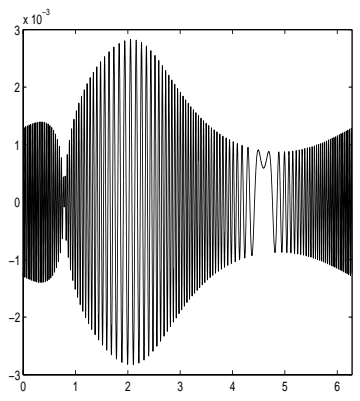
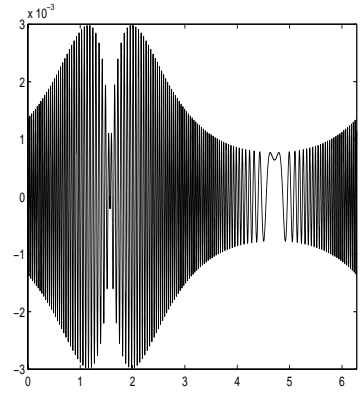
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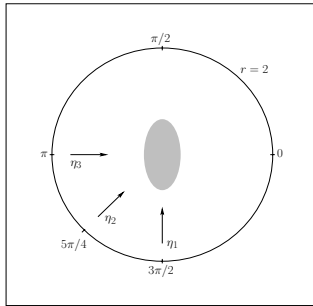
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1



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1

