Electromagnetic imaging: Representation formulas for the field perturbations caused by low volume fraction inhomogeneities

Michael S. Vogelius
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ANSWER 1: – Yes, if A is **isotropic**, i.e., if $A(x) = \gamma(x)I$.

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ANSWER 2: – Only very partially (up to a "pullback" by a diffeomorphism) if A is **anisotropic**. Sylvester (1990), Lee-Uhlmann (1987)["positive"], Kohn-Vogelius, Tartar (1987)["negative"].

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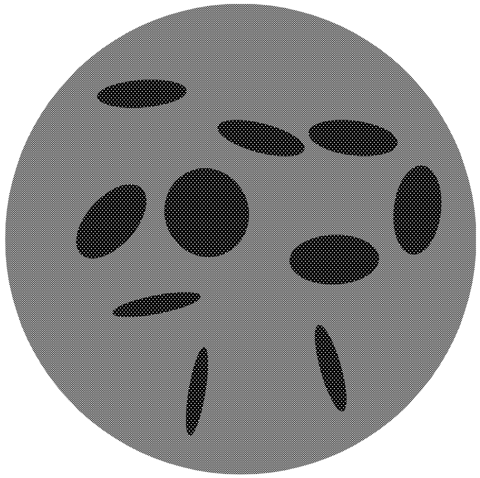
Collaborators: H. Ammari, E. Beretta, M. Brühl, Y. Capdeboscq, E. Francini, A.Friedman, M. Hanke, D. Hansen, S. Moskow, D. Volkov ....

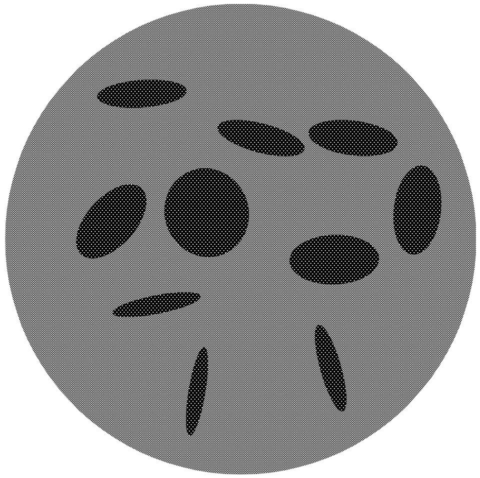


$$\gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & \text{in } \Omega \setminus \omega_\epsilon \\ \gamma_1(x) & \text{in } \omega_\epsilon \end{cases}$$

$$\omega_\epsilon \subset K_0 \subset \Omega$$

$$0 < \gamma_0(x), \gamma_1(x) < \infty$$

$$\begin{cases} \nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \phi & \text{on } \partial\Omega. \end{cases}$$



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Goal: Find an asymptotic expression for $(u_\epsilon - u_0)\big|_{\partial\Omega}$ that can be used to determine ω_ϵ (for $|\omega_\epsilon|$ small).

General Representation Formula

After the extraction of a subsequence:

$$\forall y \in \partial\Omega,\ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega \qquad\qquad + o(|\omega_\epsilon|)$$

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$N(x,y)$ is the Neumann function for $\nabla \cdot (\gamma_0 \nabla\ )$:

$$\nabla_x \cdot \big(\gamma_0 \nabla_x N(x,y)\big) = \delta_y \text{ in } \Omega$$

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The probability measure $\mu = \lim_{|\omega_\epsilon| \to 0} \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon}$ converges weak* in the dual of $C^0\left(\overline{\Omega}\right)$.

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After the extraction of a subsequence:

$$\forall y \in \partial\Omega, \; u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)$$

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M is a matrix valued function in $L^2(\Omega, d\mu)$. The values of M are symmetric, positive definite matrices.

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$$\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \nabla u_\epsilon dx \rightharpoonup M \nabla u_0 d\mu$$

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$$\text{and } \|\nabla (u_\epsilon - u_0)\|_{L^2(\Omega)} \leq C |\omega_\epsilon|^{\frac{1}{2}} \quad (u_0 \text{ is smooth}).$$

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But, notice that $v_0^j = x_j + cst$ is not a solution to

$$\begin{cases} \nabla \cdot (\gamma_0 \nabla v_0) = 0 & \text{in } \Omega \\ \gamma_0 \frac{\partial v_0}{\partial \nu} = \psi & \text{on } \partial\Omega. \end{cases}$$

unless γ_0 is a constant!

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Instead, construct

$$\begin{cases} \nabla \cdot \left(\gamma_\epsilon \nabla v_\epsilon^j\right) = \nabla \cdot \left(\gamma_0 \nabla v_0^j\right) & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial v_\epsilon^j}{\partial \nu} = \gamma_0 \frac{\partial v_0^j}{\partial \nu} & \text{on } \partial\Omega, \\ v_0^j = x_j + \text{cst}, & \end{cases}$$

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and define

$$\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial v_\epsilon^j}{\partial x_k} \xrightarrow{\text{def}} M_{jk} d\mu.$$

$$\text{To prove } \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \rightharpoonup M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

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$$\|$$

$$\tfrac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx$$

$$\text{To prove } \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \rightharpoonup M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

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$$\tfrac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx \qquad \tfrac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx$$

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$$\| \qquad\qquad\qquad \|$$

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$$\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

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\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j dx & = & \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j dx \\
\| & & \| \\
\frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx & & \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx \\
\downarrow & & \downarrow \\
\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu & & \int (\gamma_0 - \gamma_1) \lim_{\epsilon \to 0} \left(\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \right) dx
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+compensated compactness (judicious integration by parts)

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\frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx & & \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx \\
\downarrow & & \downarrow \\
\displaystyle\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu & & \displaystyle\int (\gamma_0 - \gamma_1) \lim_{\epsilon \to 0} \left(\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \right) dx
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+ cut-offs.

Bounds and variational characterization of M

$$\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j d\mu \quad = \quad \frac{1}{|\omega_\epsilon|} \min_{w\in H^1(\Omega)} \int_\Omega \gamma_\epsilon \left|\nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon}\xi\right|^2 dx$$

$$+\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon} (\gamma_1 - \gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,dx \ + o(1)\ .$$

Bounds and variational characterization of M

$$\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \phi d\mu \;=\; \frac{1}{|\omega_\epsilon|} \min_{w \in H^1(\Omega)} \int_\Omega \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi dx$$
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for all uniformly positive, smooth functions ϕ on Ω.

Bounds and variational characterization of M

$$\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \phi d\mu \quad = \quad \frac{1}{|\omega_\epsilon|} \min_{w \in H^1_{per}(Q)} \int_Q \gamma_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi dx$$
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for all uniformly positive, smooth functions ϕ on Q, $\Omega \subset Q$.

Bounds and variational characterization of M

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for all uniformly positive, smooth functions ϕ on Q, $\Omega \subset Q$.

This yields

$$\min\left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2 \le M(x)\xi\cdot\xi \le \max\left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2 \, ,$$

for $\xi \in I\!R^n$, μ almost everywhere in the set $\{x \in \Omega : \gamma_0(x) \neq \gamma_1(x)\}$.

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Bounds and variational characterization of M

$$
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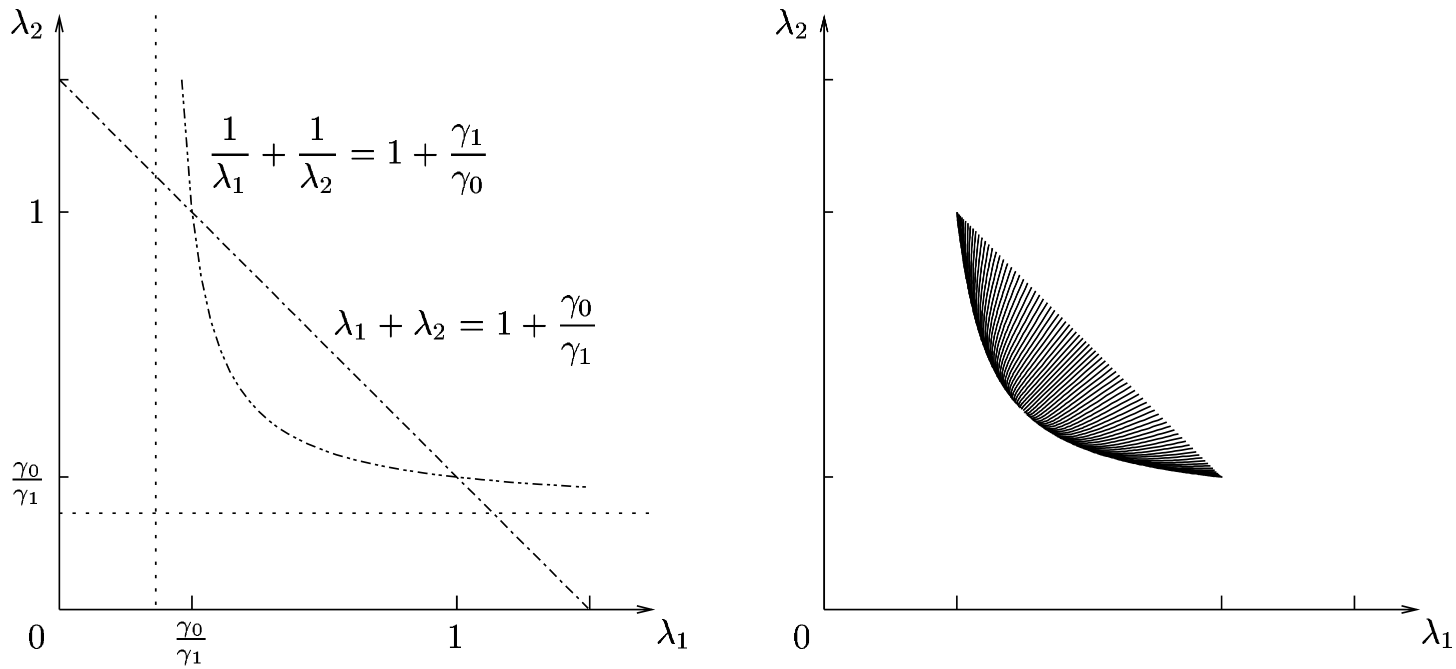

Bounds

M satisfies

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Three of these bounds are attained for a single "sheet-like" inclusion. In that case the polarization egenvalues "parallel" to the sheet are 1, and the eigenvalue across the sheet is $\frac{\gamma_0}{\gamma_1}$. The fourth bound is attained for a single inclusion in the shape of a ball.

Applications

$$\forall y \in \partial\Omega,\ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)$$

May be used :

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May be used :

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).

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May be used :

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).

2. To estimate the volume of inhomogeneities of moderate size (Capdeboscq, MV)

Detecting locations

Suppose $\omega_\epsilon = \cup_{j=1}^{p}(z_j + \epsilon B_j)$ (the inhomogeneities "shrink" to points z_j).

$$
\begin{aligned}
D(\phi)(\cdot) &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, \cdot) d\mu(x) \\
&= |\omega_\epsilon| \sum_{j=1}^{p} (\gamma_1 - \gamma_0) \alpha_j M^j \nabla u_0(z_j) \cdot \nabla_x N(z_j, \cdot)
\end{aligned}
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Is linear in in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np).

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Is linear in in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np). In fact,

$$\mathcal{R}(D) = \text{span}\{e_k \cdot \nabla_x N(z_j, \cdot)|_{\partial B} : k = 1, \ldots, n, j = 1, \ldots, p\}.$$

Probe with $g_{z,d} = d \cdot \nabla_x N(z, \cdot)|_{\partial B}$. Then $g_{z,d} \in \mathcal{R}(D)$ iff $z \in \{z_j : j = 1, \ldots, p\}$. Note also that $\mathcal{R}(D)$ is well approximated by $\mathcal{R}(\Lambda_\epsilon - \Lambda_0)$ (the measured Neumann-Dirichlet "difference" map).

Detecting locations

Method:

Detecting locations

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1. Compute the SVD decomposition of $\Lambda_\epsilon - \Lambda_0$, and the projector onto the space spanned by the first m eigenvectors, P_m.

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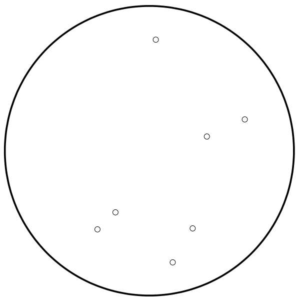
2. For a test point z, compute

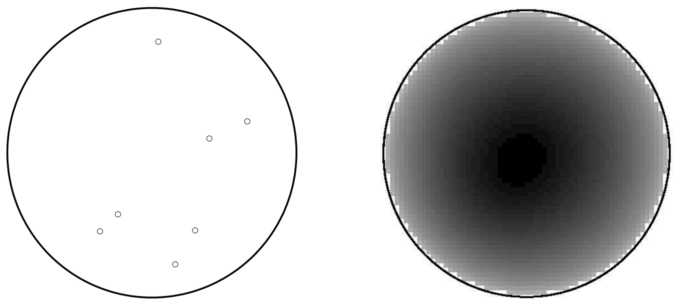
$$\cot\,\theta_m(z) = \frac{\|P_m g_{z,d}\|}{\big\|(I - P_m) g_{z,d}\big\|}.$$

For $m = pn$, $z \in \{z_j : j = 1, \ldots, p\} \Leftrightarrow \cot\,\theta_m(z) = \infty$.

Detecting locations



Detecting locations



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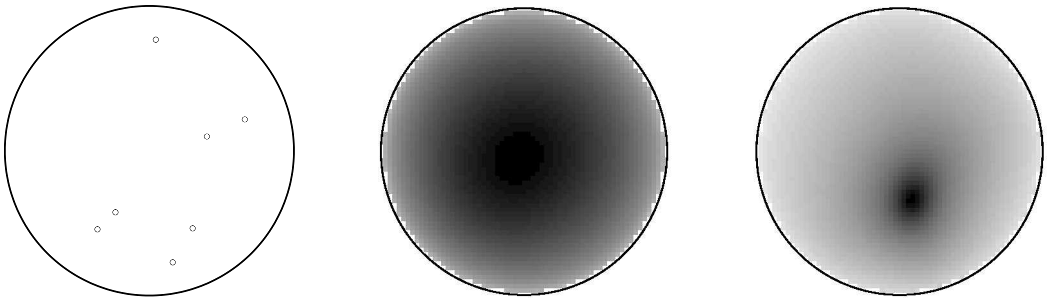

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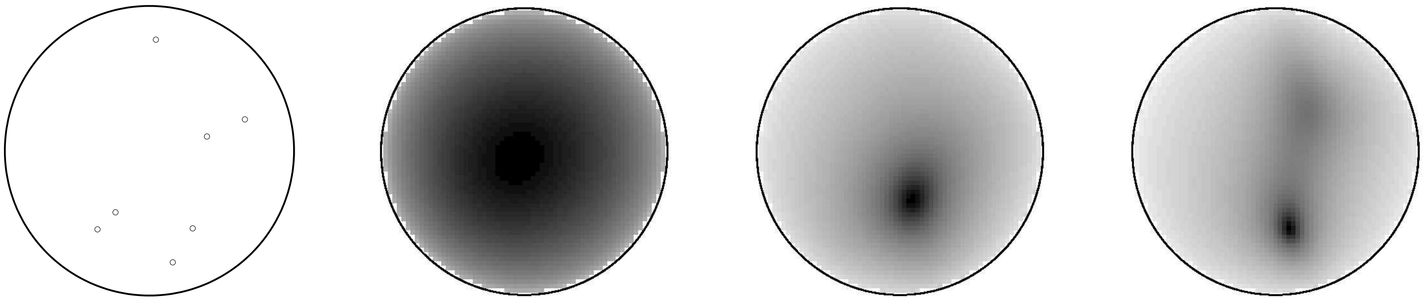

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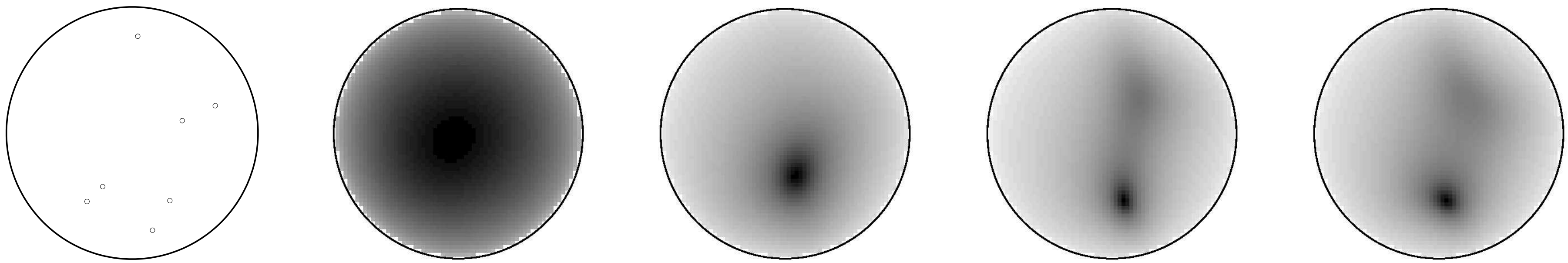

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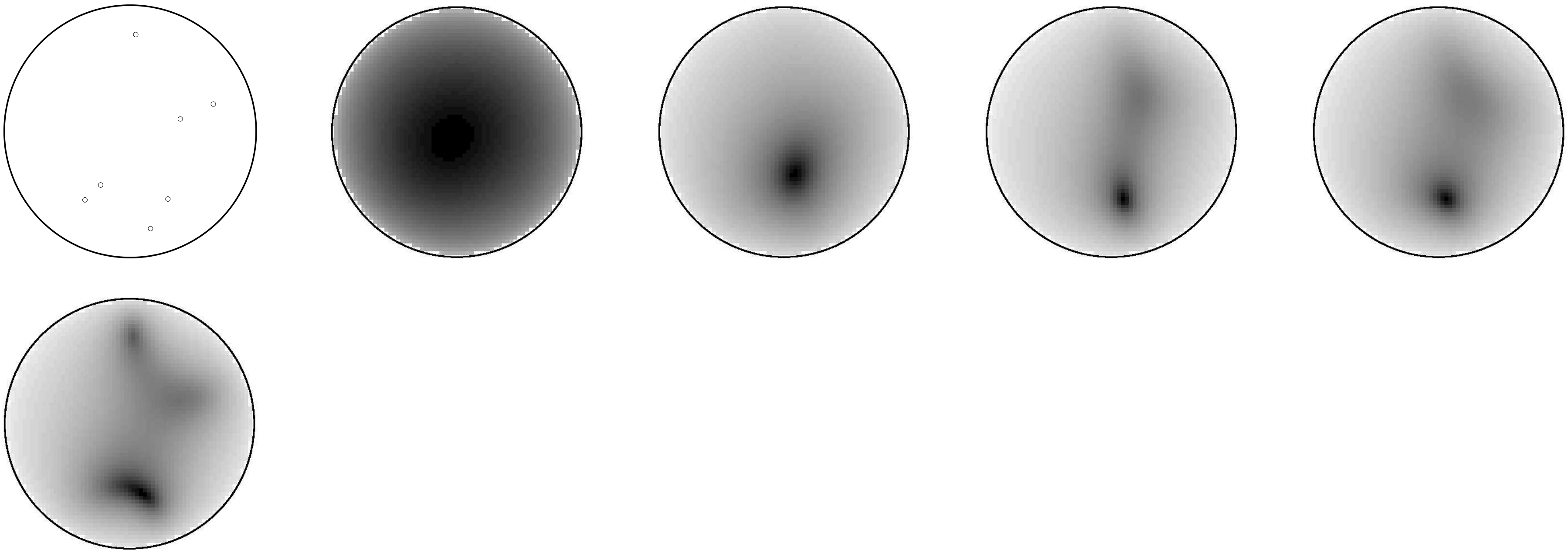

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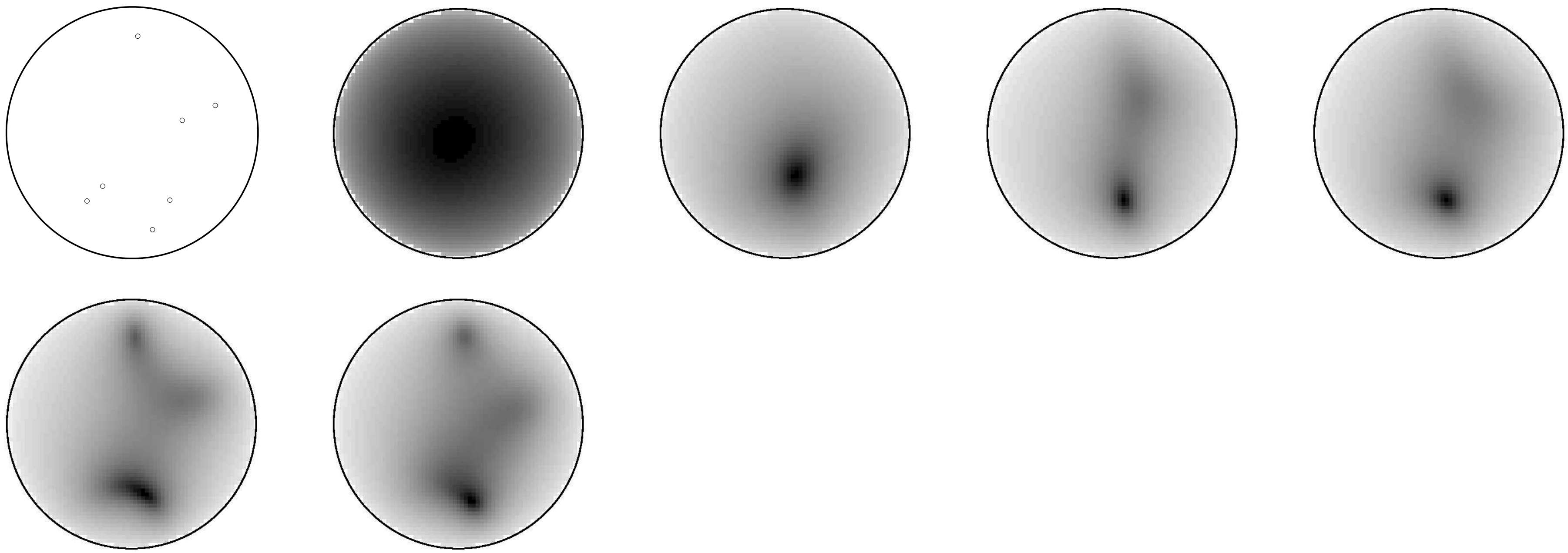

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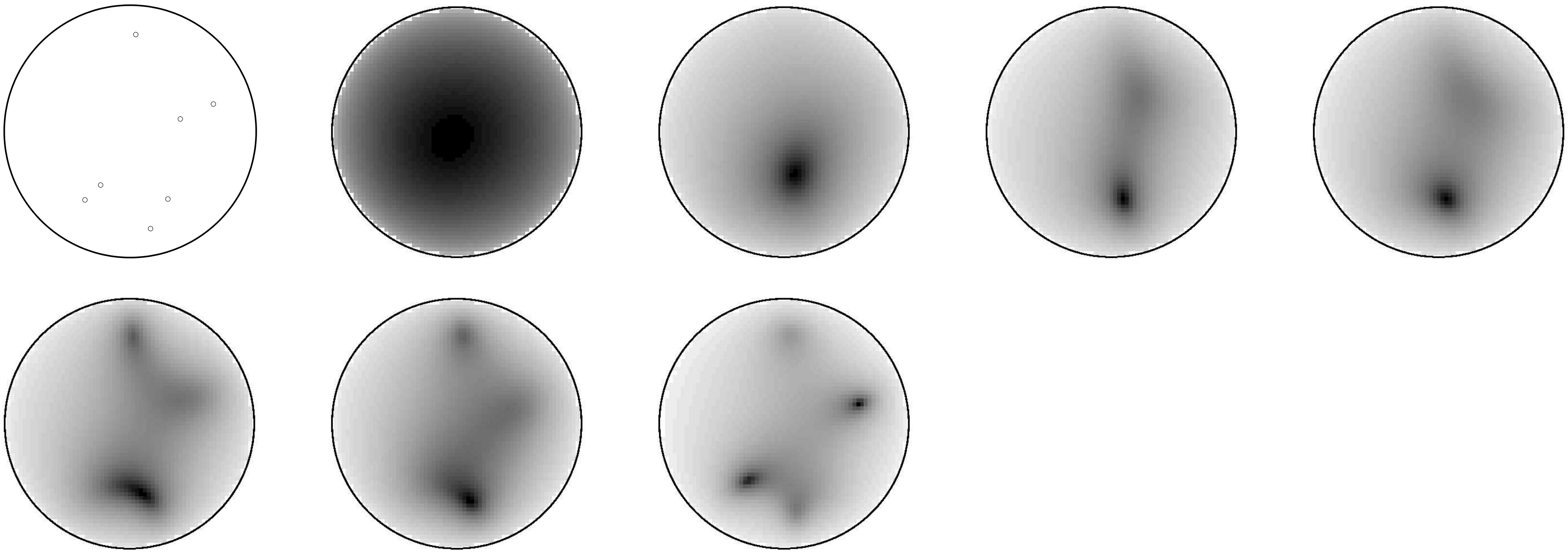

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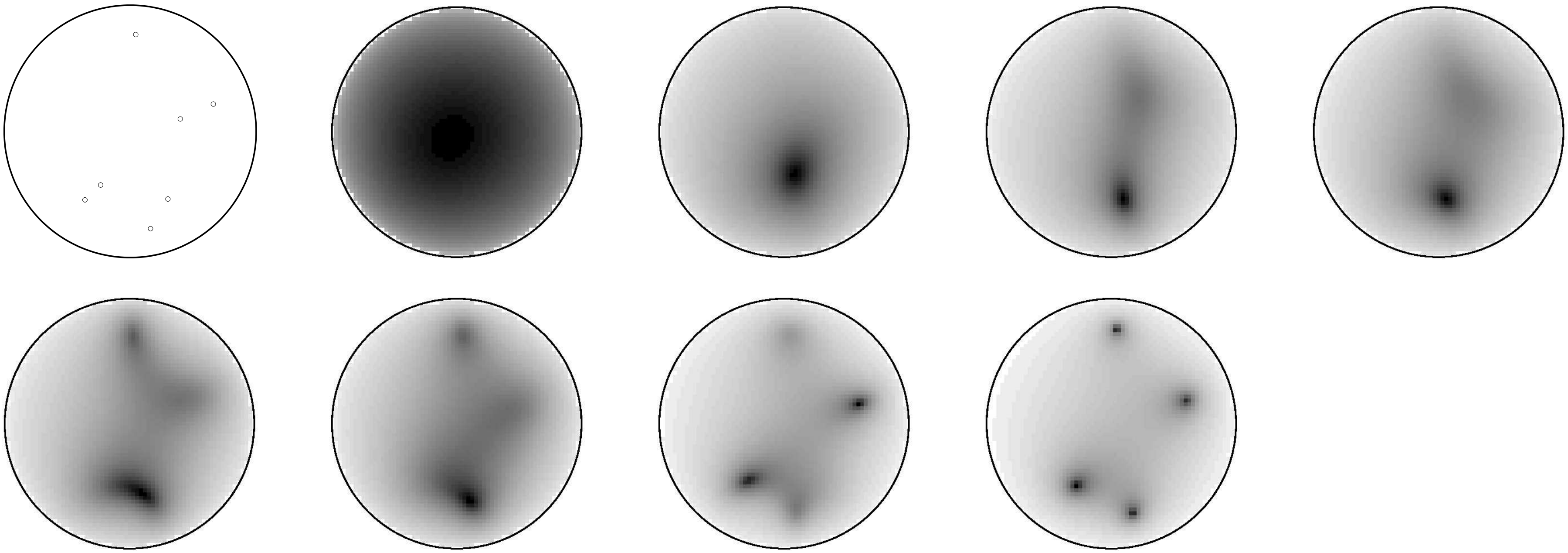

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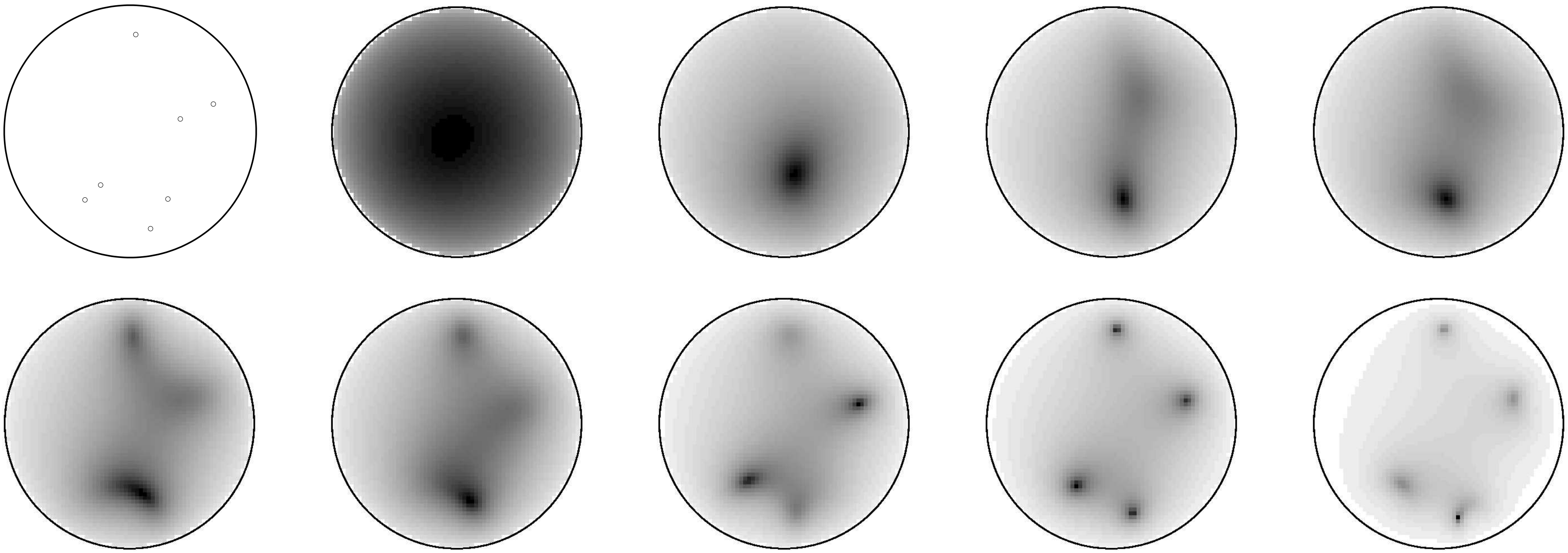

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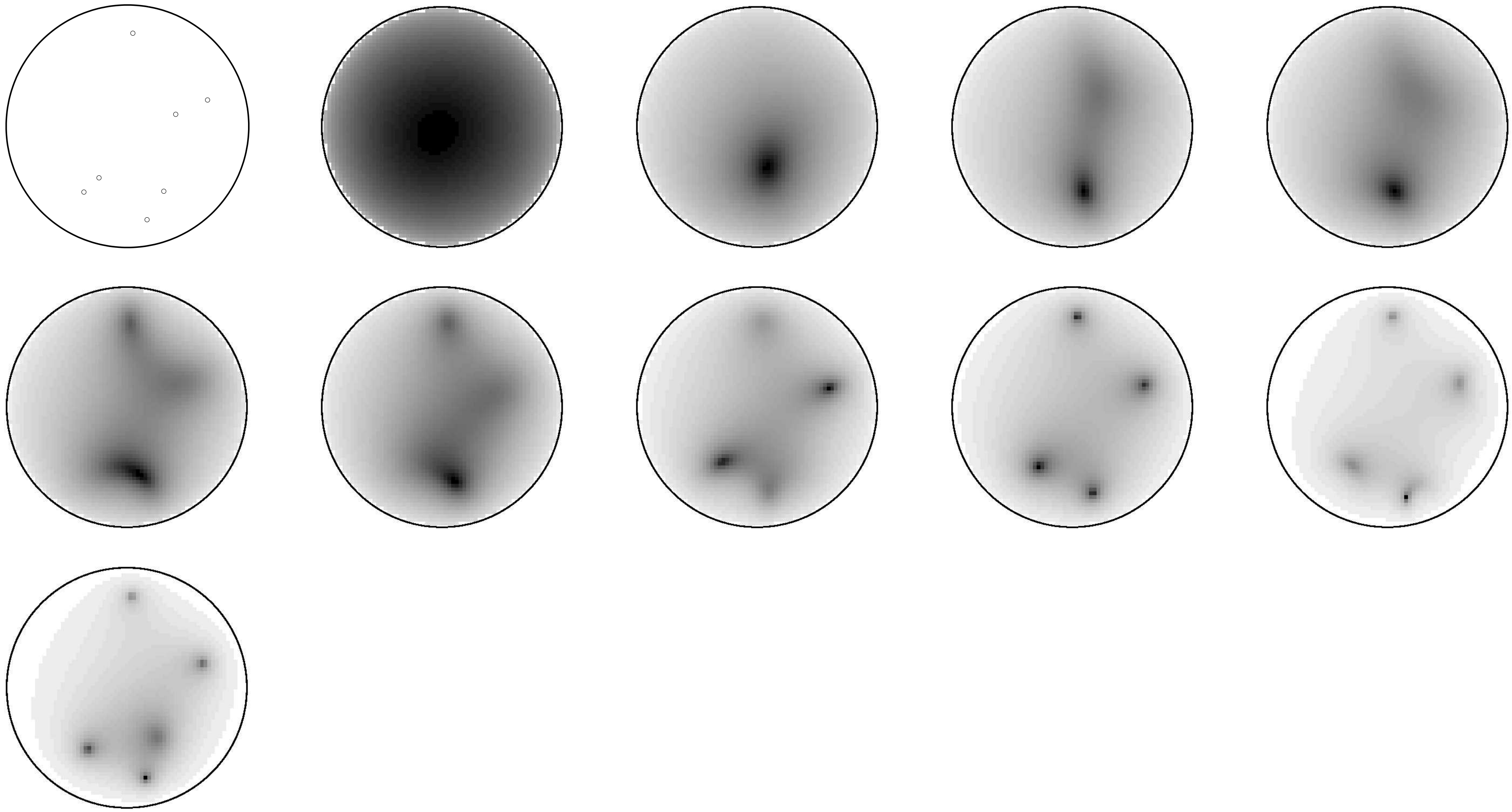

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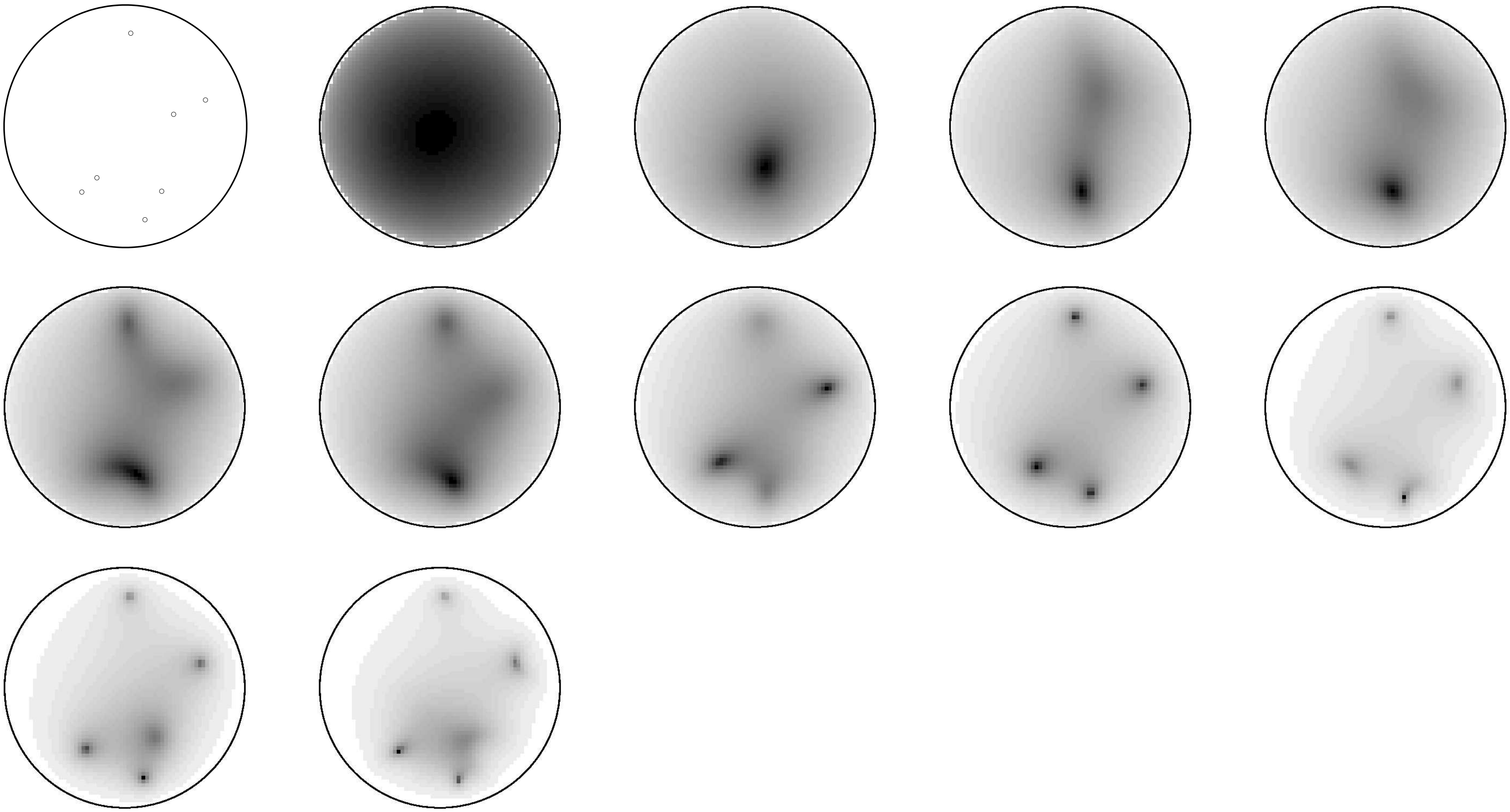

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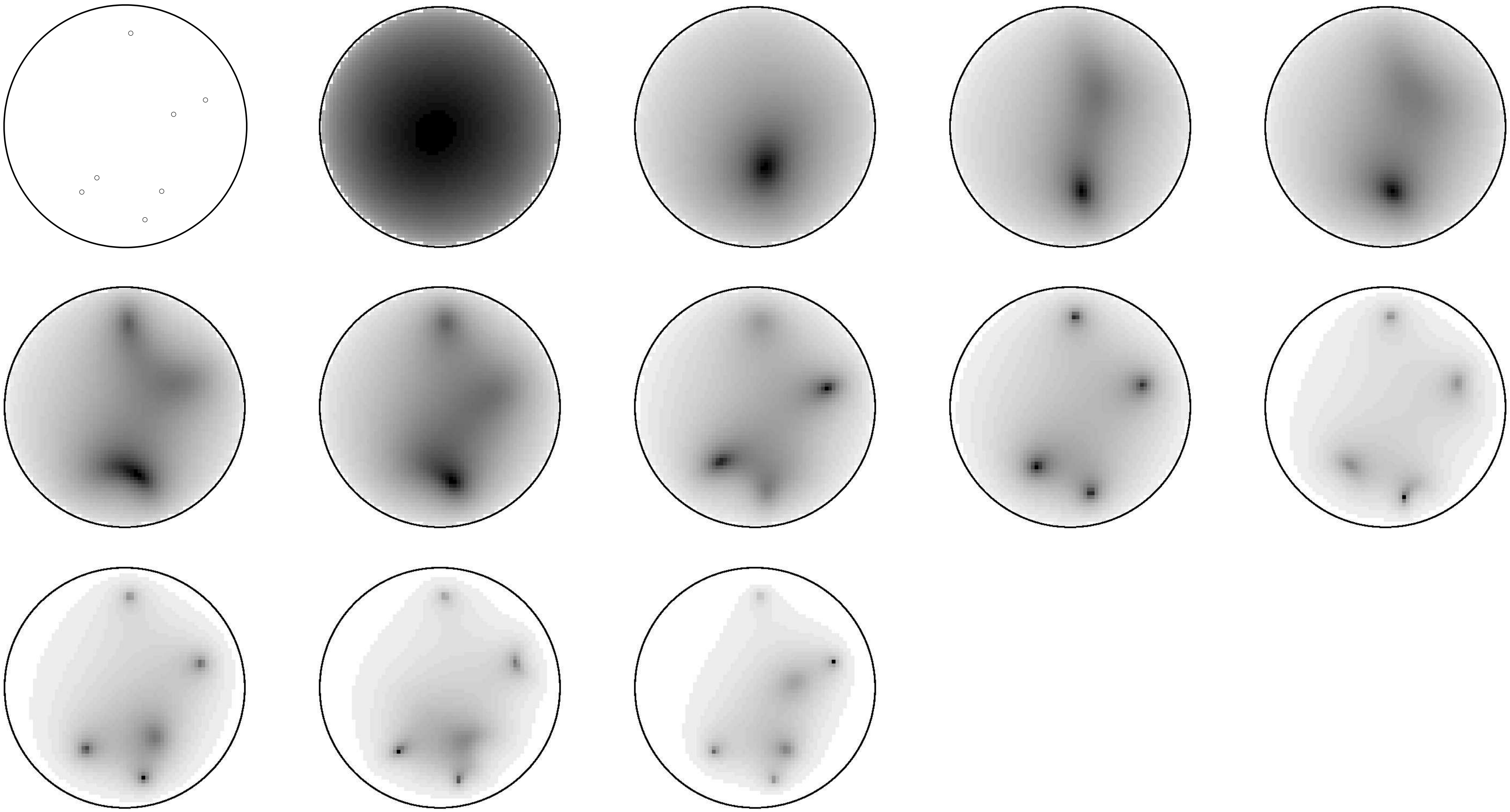

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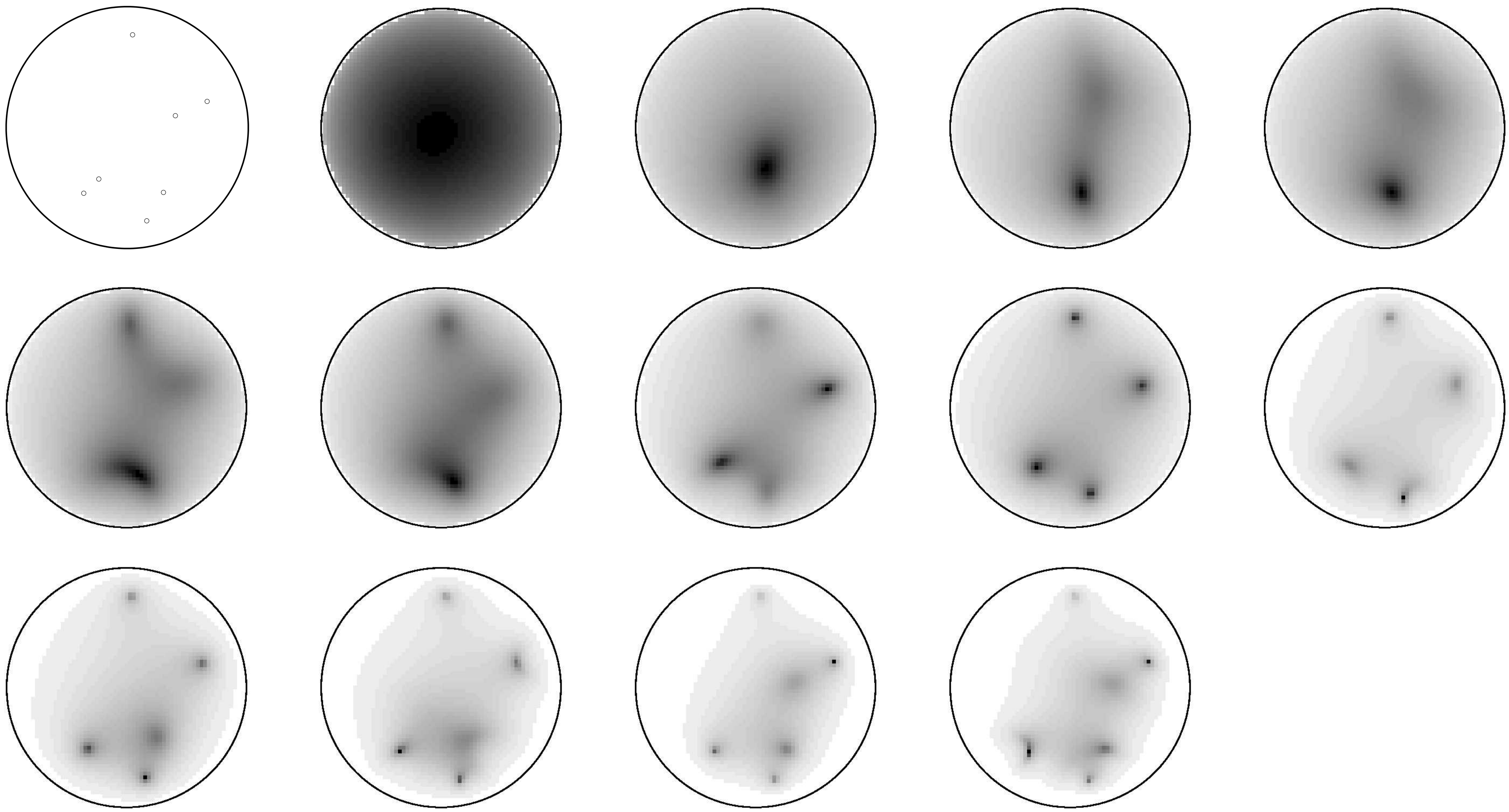

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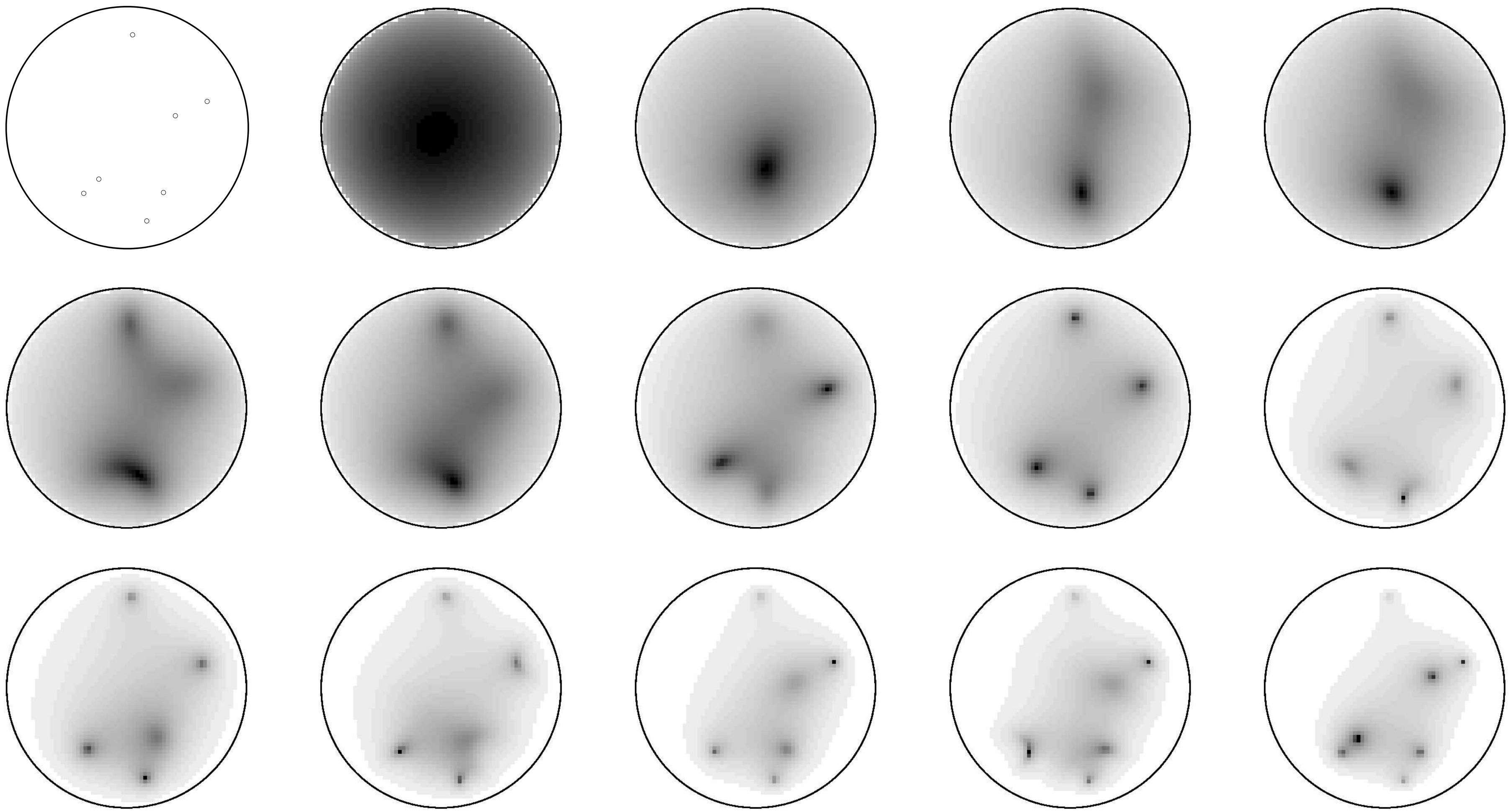

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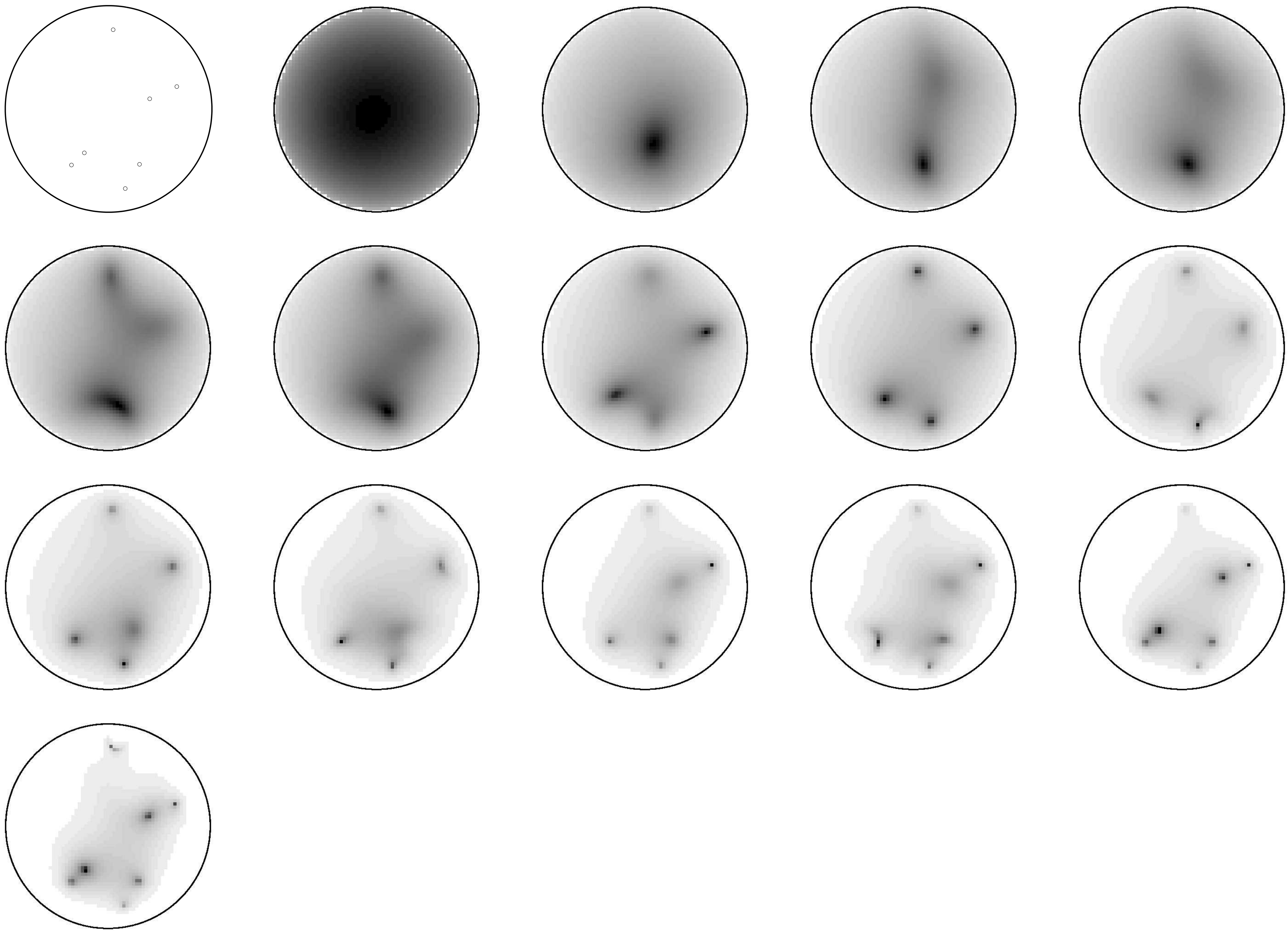

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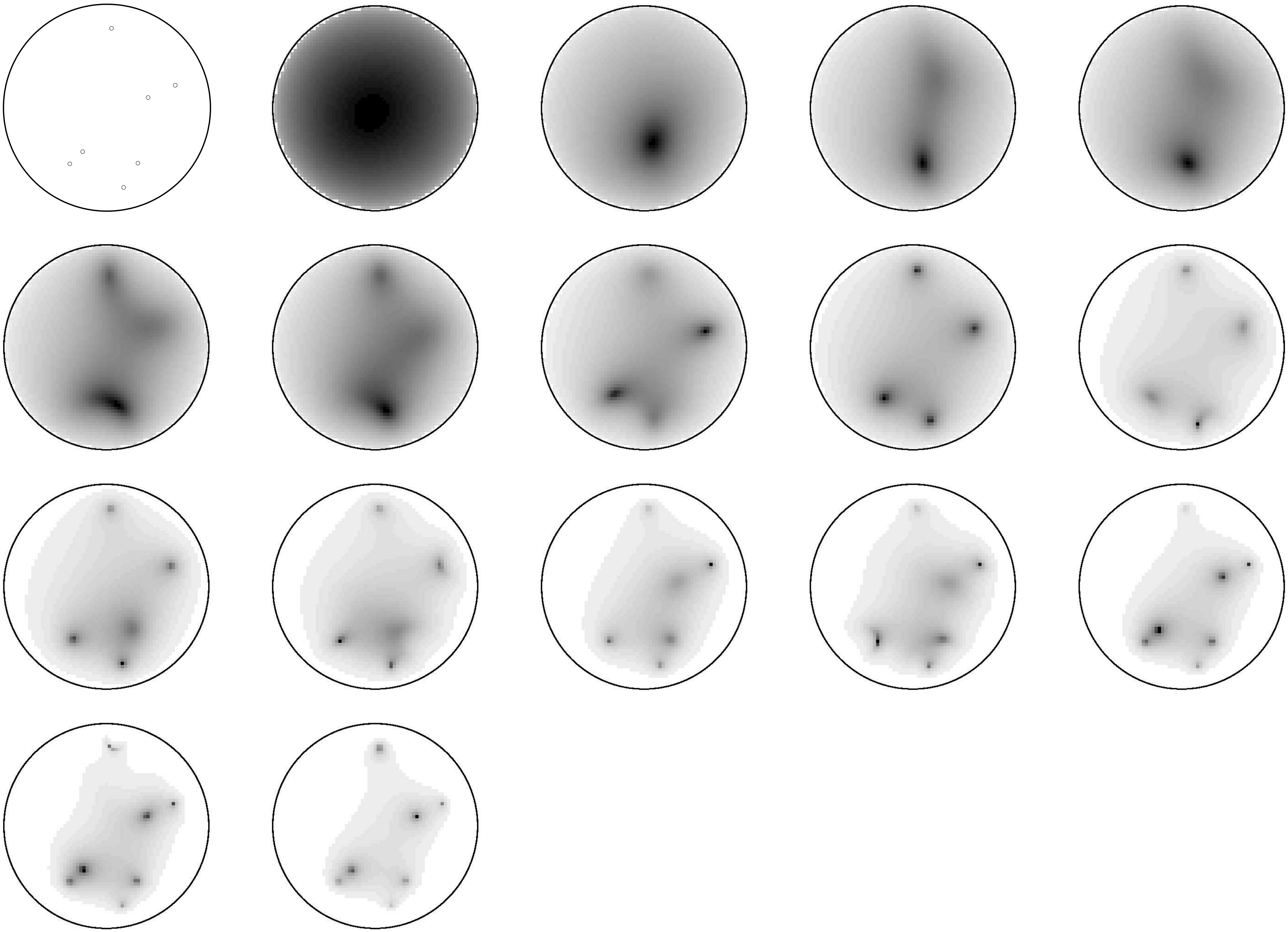

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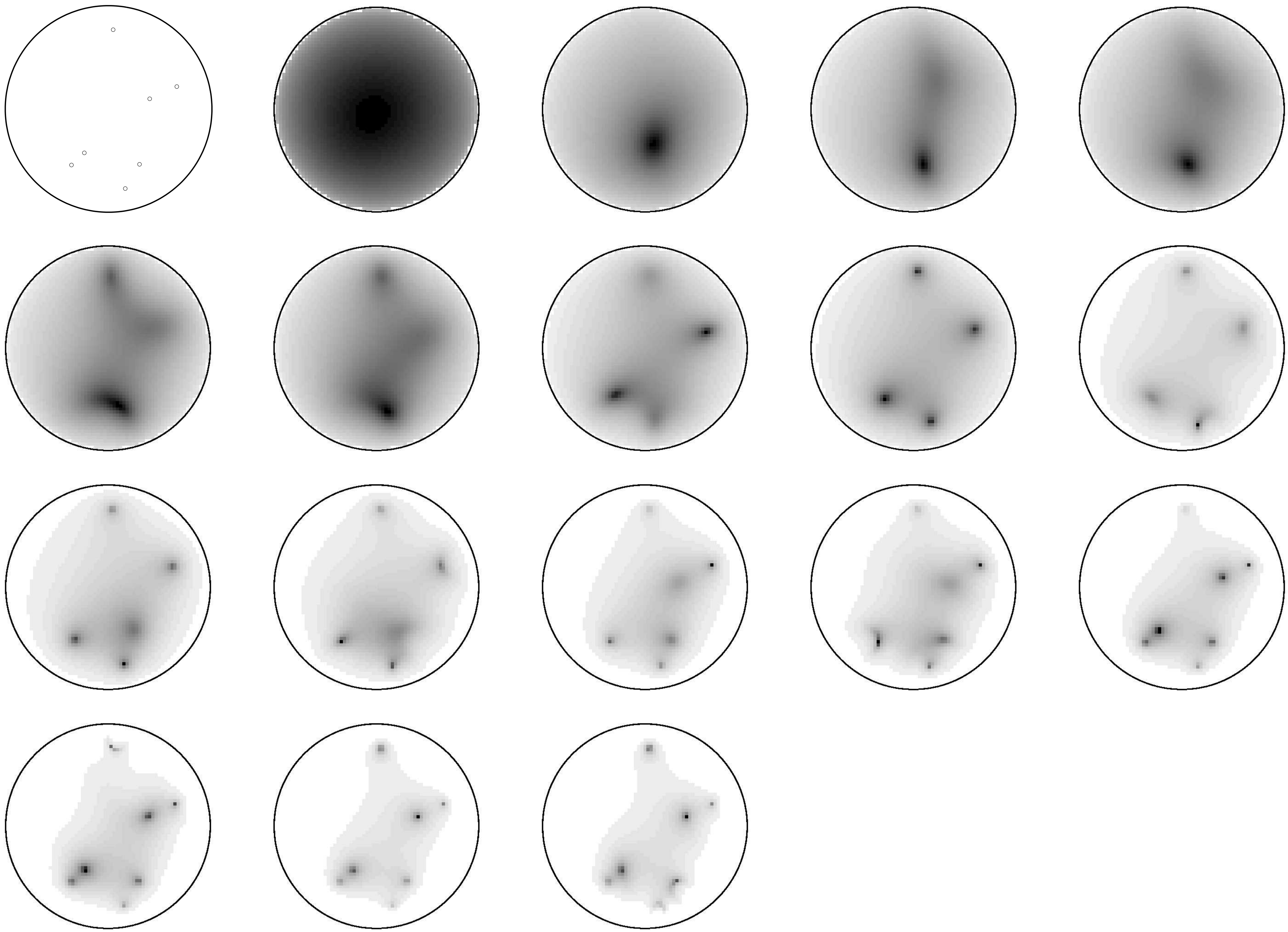

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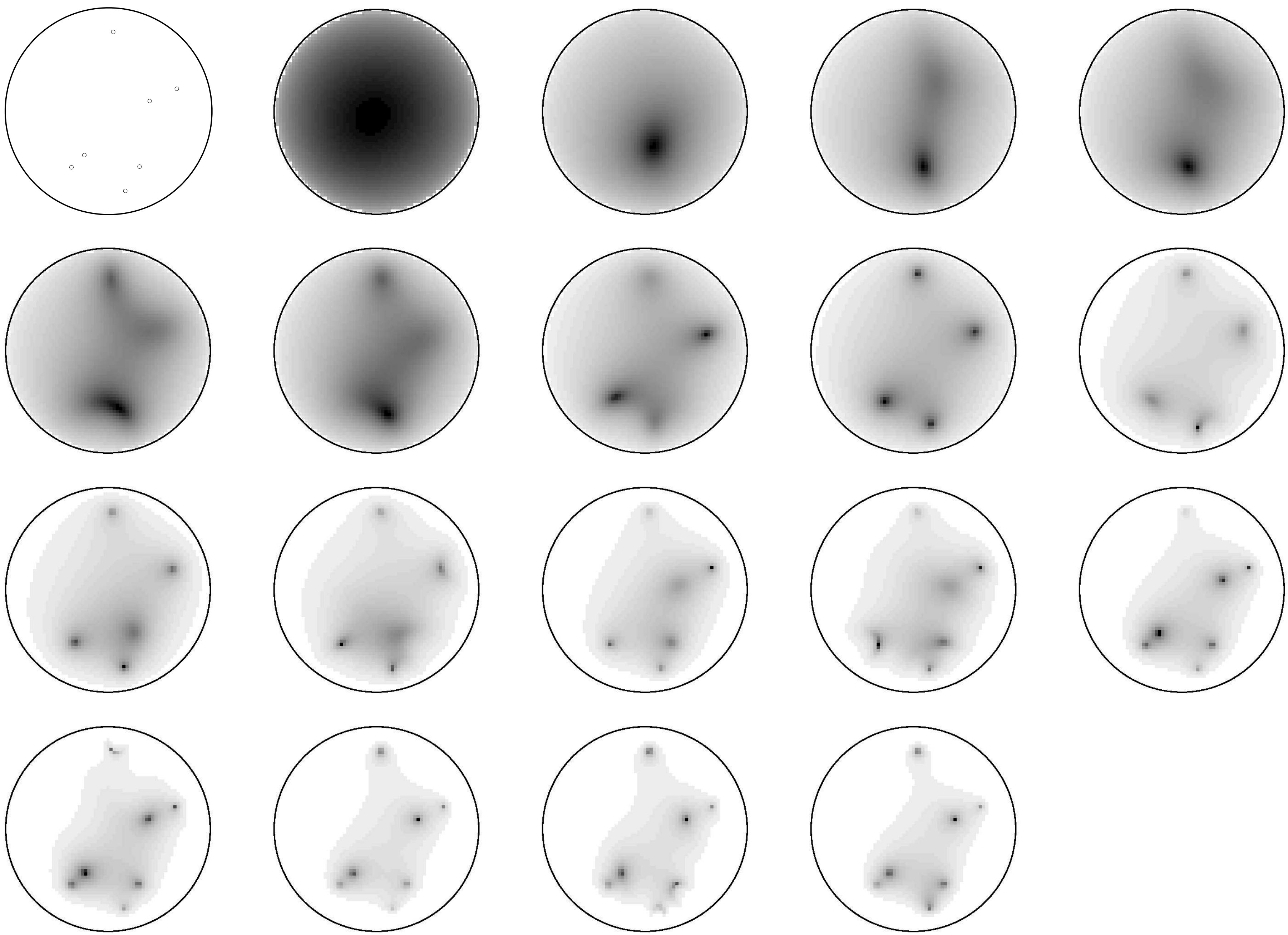

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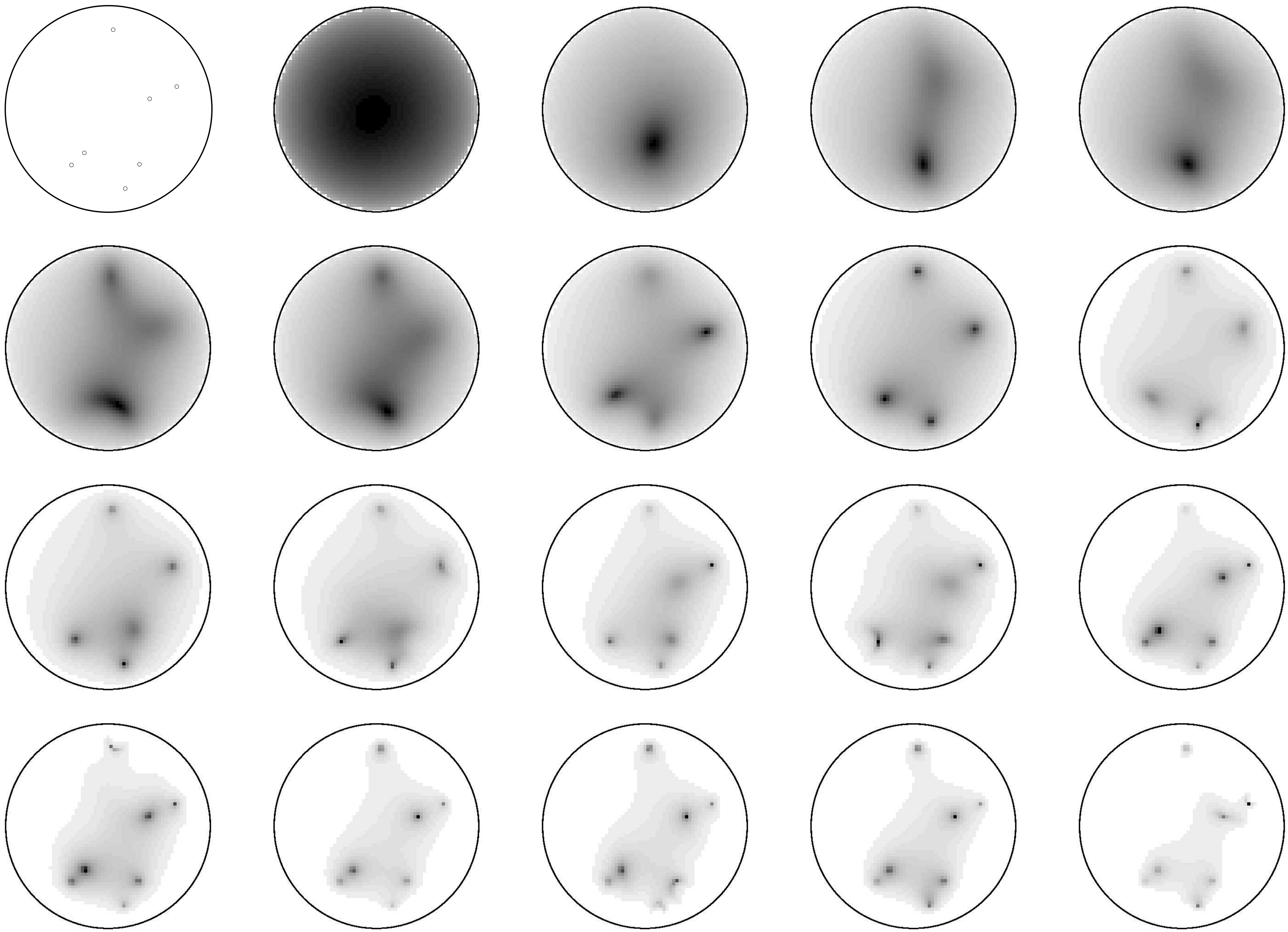

Volume estimation

Suppose γ_0 and γ_1 are constants.

$$u_\epsilon(y) - u_0(y) \quad = \quad |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x)$$

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Pick $\nabla u_0 = e_j$, $j = 1, \ldots, n$

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$$\left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \int_\Omega M_{jj} d\mu, \text{ and } \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \text{trace}(\int_\Omega M d\mu).$$

Volume estimation

Using the bounds on M we obtain

$$\min\left(1, \frac{\gamma_1}{\gamma_0}\right)\left|\frac{\mathrm{data}_j}{\gamma_1 - \gamma_0}\right| \leq |\omega_\epsilon| \leq \max\left(1, \frac{\gamma_1}{\gamma_0}\right)\left|\frac{\mathrm{data}_j}{\gamma_1 - \gamma_0}\right|.$$

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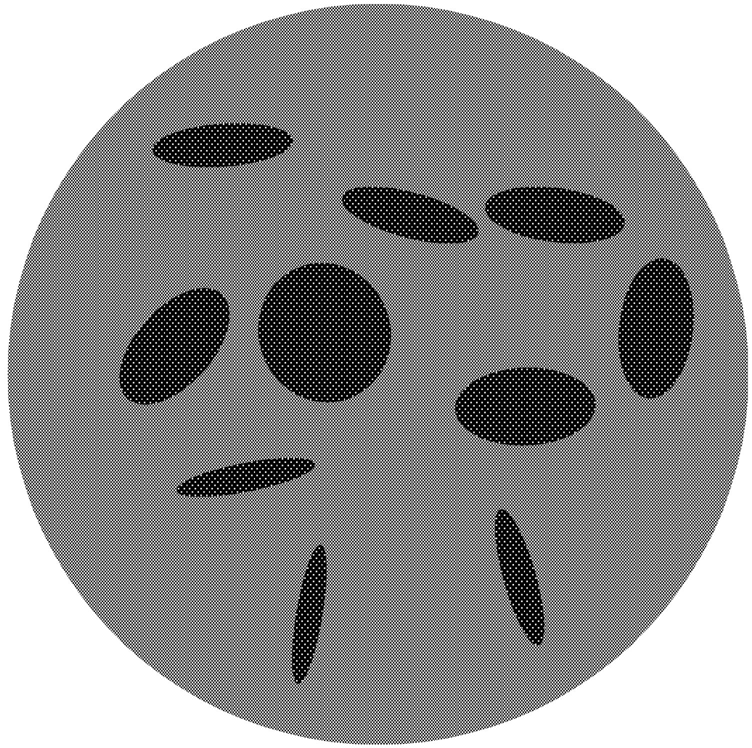
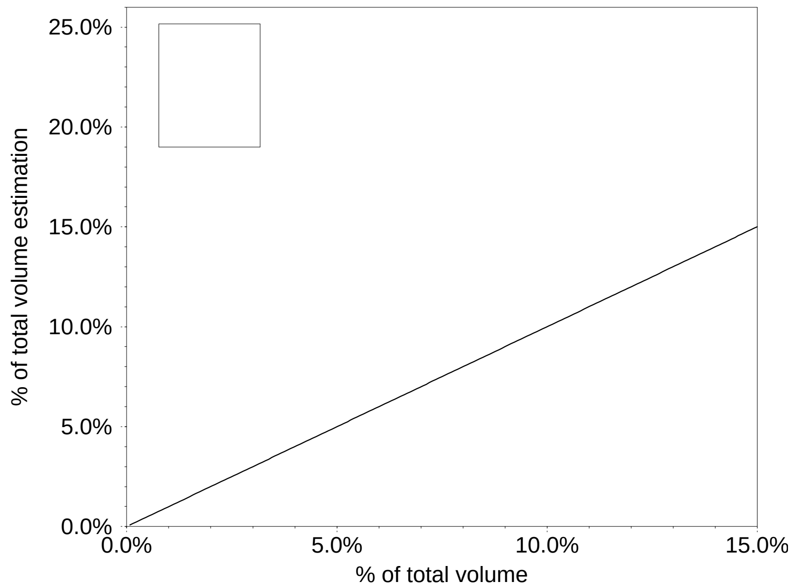
n measurement bounds. Only asymptotic bounds.

Random Ellipses



Volume estimation, with $\gamma_0 > \gamma_1$



Proportion of total volume occupied by the inhomogeneities.

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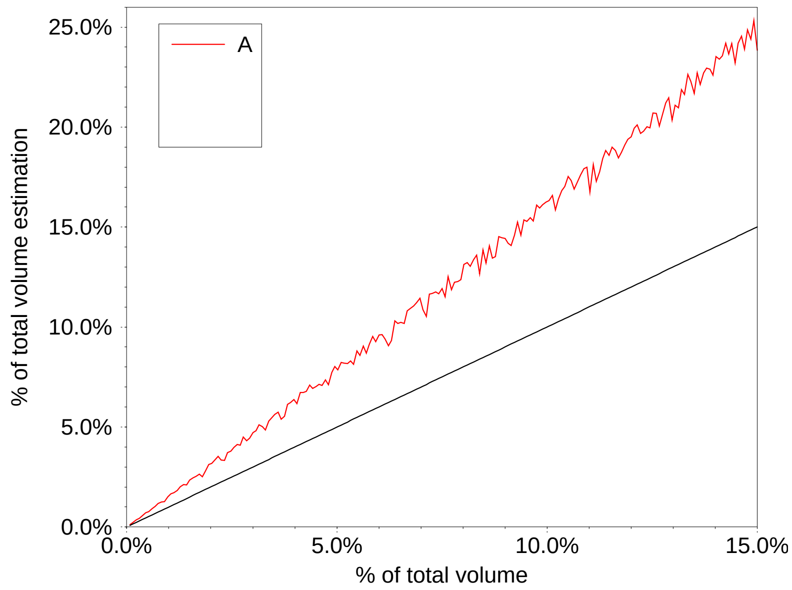

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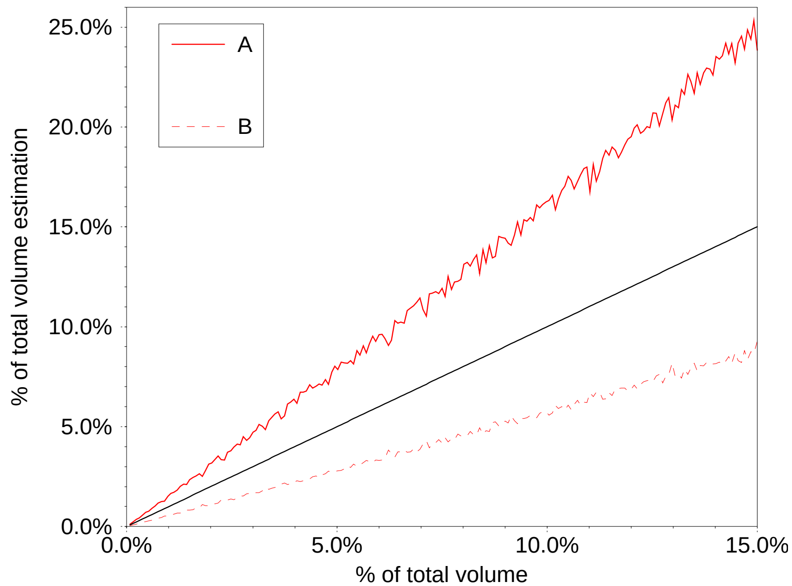

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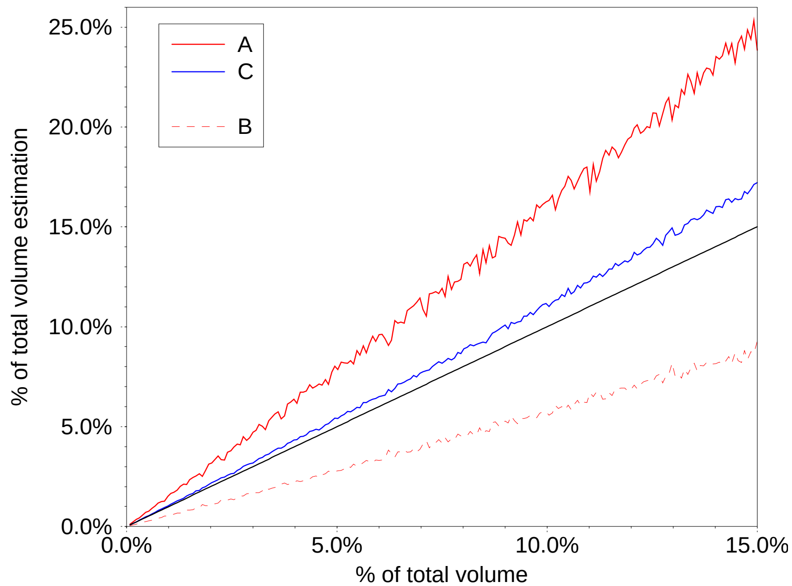

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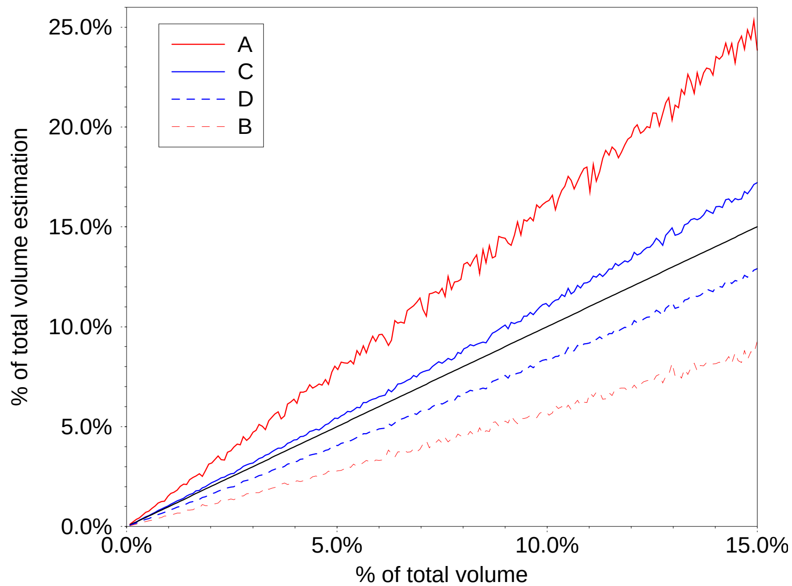

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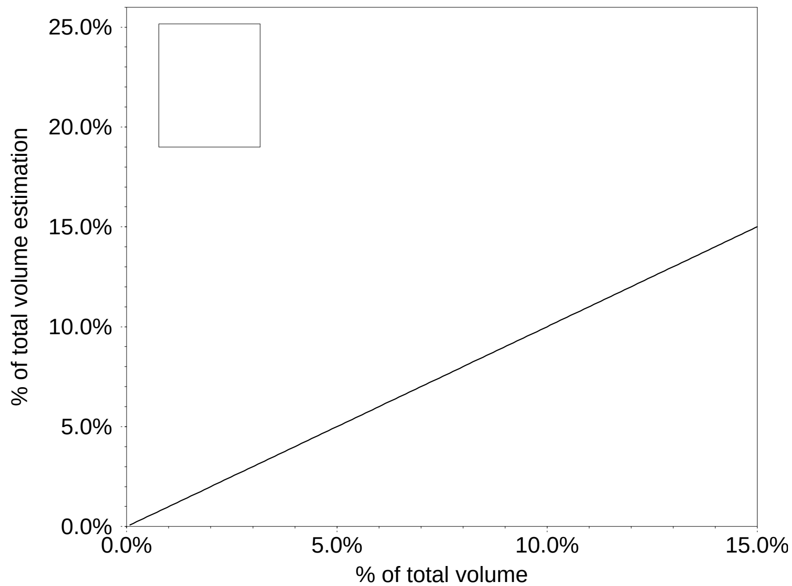

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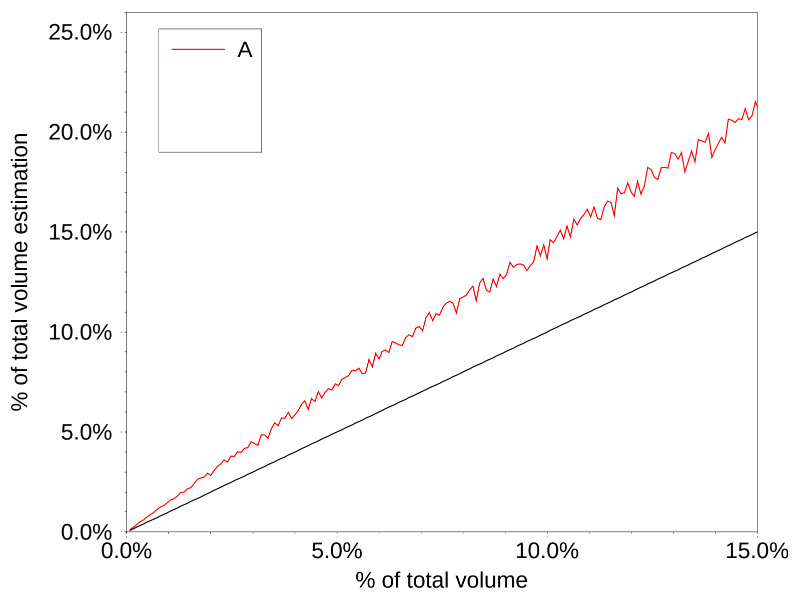

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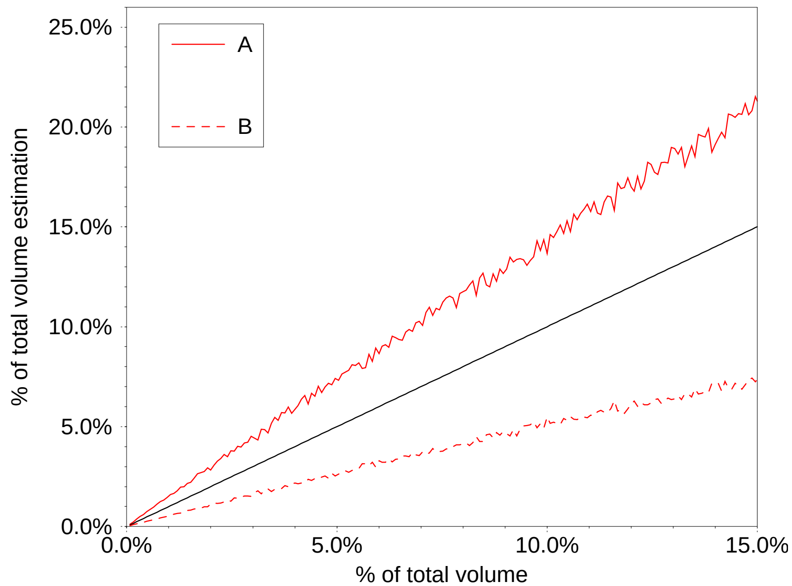

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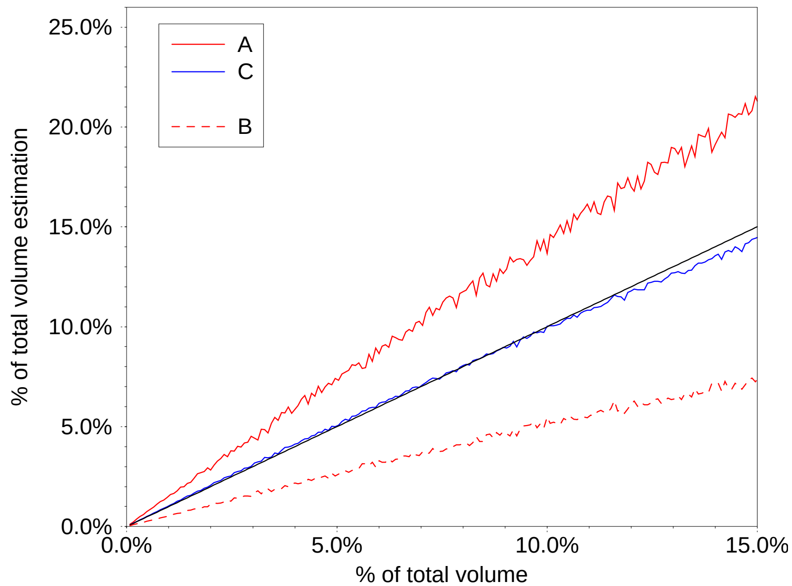

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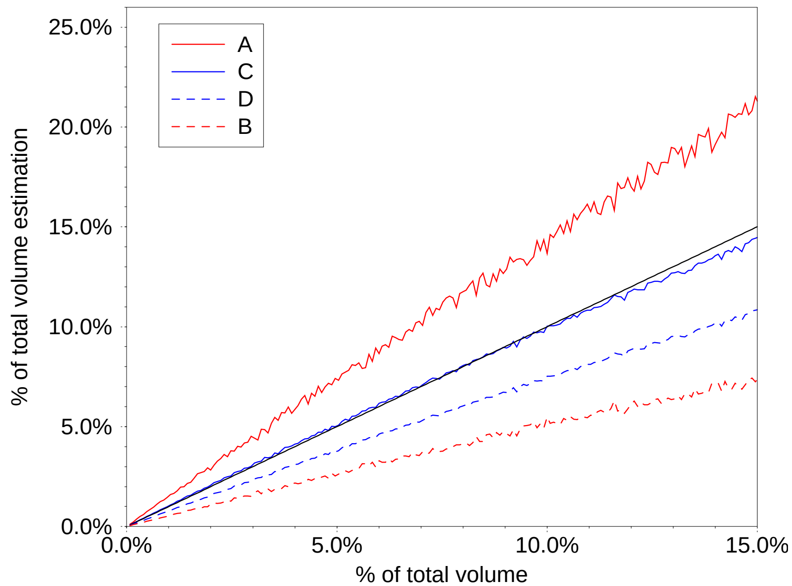

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We now take E_0 to be a plane wave, *i.e.*,

$$\Phi^k(x,y) = -\tfrac{i}{4} H_0^{(1)}(k\sqrt{\mu_0 q_0}|x-y|) \quad \text{and} \quad E_0(x) = e^{ik\sqrt{\mu_0 q_0}\xi \cdot x}$$

The first two terms are now of order $(\epsilon k)^2/(1+\sqrt{k})$ and it is possible to prove that the remainder is of order $o((\epsilon k)^2)/(1+\sqrt{k})$, as $\epsilon k \to 0$, $k \geq k_0 > 0$.

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In other words, "our small inhomogeneity approximation" holds uniformly as $\epsilon k \to 0$, and *not just* for fixed k as $\epsilon \to 0$.

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We now show the L^2 norm of $E_\epsilon(y) - E_0(y)$ on the circle $|y| = 2$ for three circular inhomogeneities of radius $\epsilon = 0.01,\ 4 \cdot 0.001$, and 0.001, respectively.

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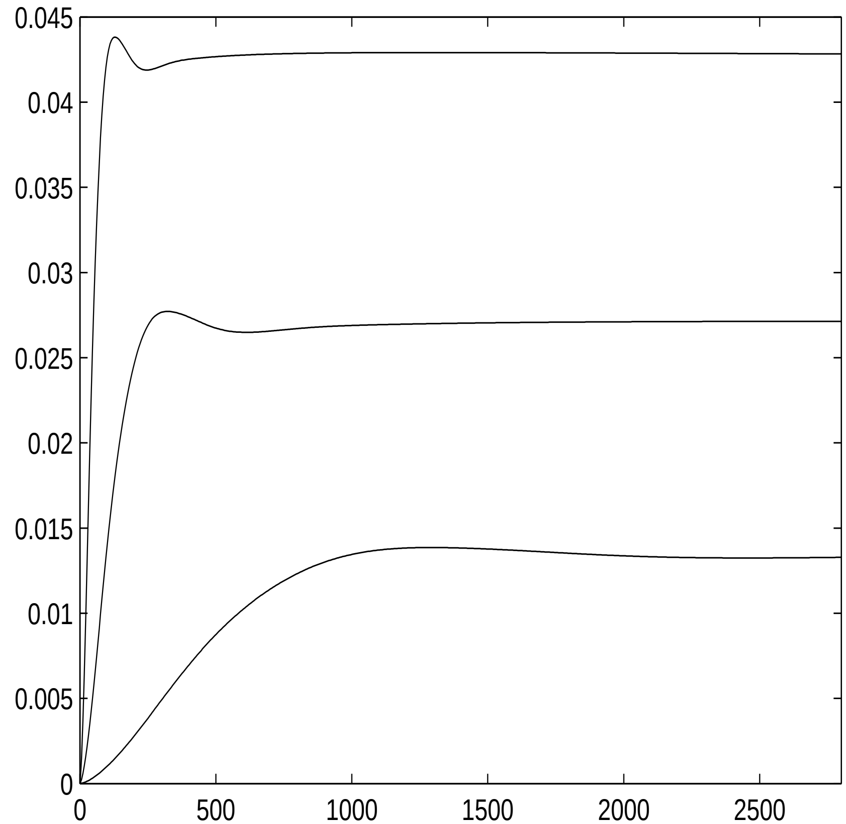

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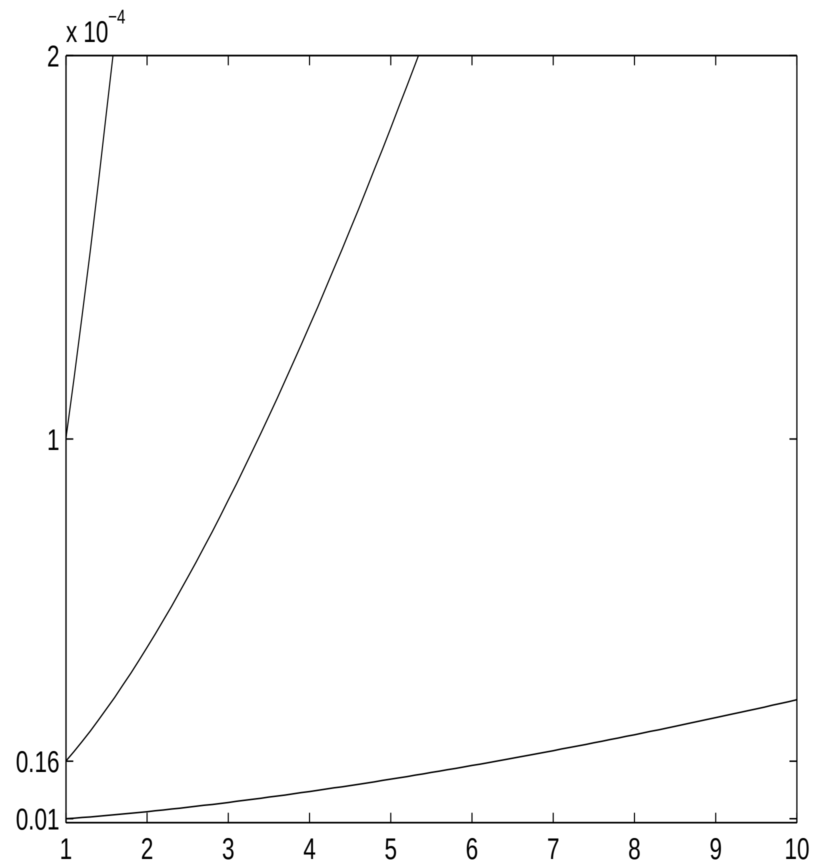

These were for a conducting inhomogeneity ($\mu_1 = 2$, $q_1 = 2 + 2i$ inside the inhomogeneity, $\mu_0 = q_0 = 1$ outside). For a non-conducting inhomogeneity a similar computation gives

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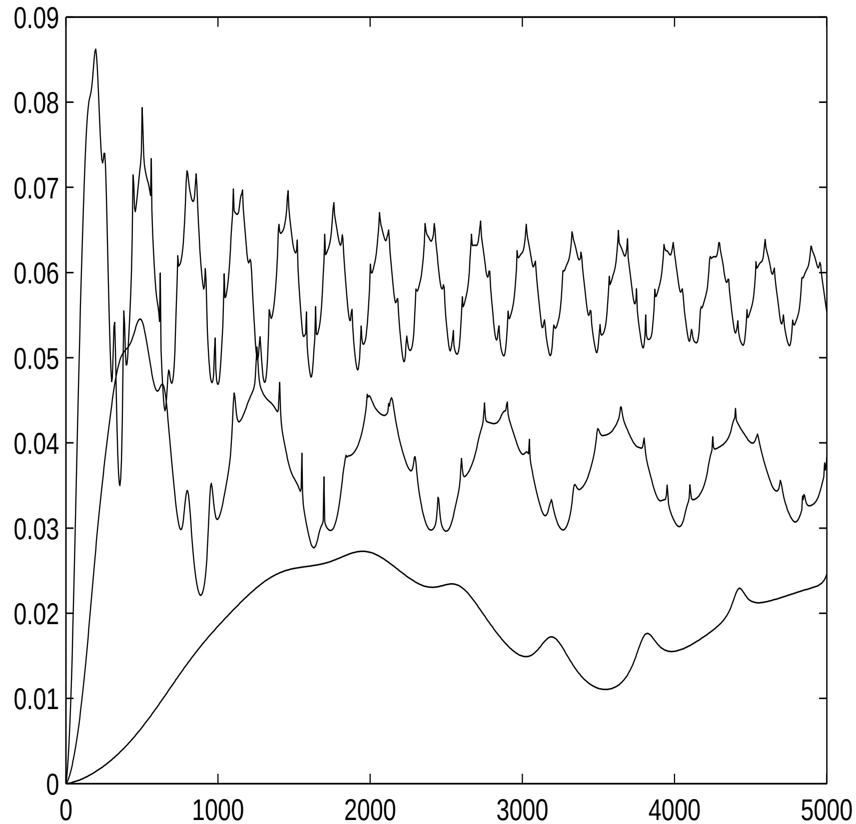

The 2d Helmholtz Equation (Transverse Magnetic Maxwell)

One inhomogeneity of the form ϵD; change variables $x \longrightarrow z = \frac{x}{\epsilon}$, $k \longrightarrow \lambda = k\epsilon$, $V_\lambda(z) = E_{k,\epsilon}(z\epsilon)$. Then

$$\nabla_z \cdot \left(\frac{1}{\mu} \nabla_z V \right) + \lambda^2 q V = 0 \quad \text{in } I\!R^2 \quad .$$

Introduce $V_\lambda^{(s)}(z) = V_\lambda(z) - e^{i\lambda\sqrt{\mu_0 q_0}\xi\cdot z}$, then

$$\begin{aligned} E_{k,\epsilon}(y) - E_{k,0}(y) &= V_\lambda^{(s)}(\frac{y}{\epsilon}) \\ &= \int_{\partial D} \frac{\partial}{\partial n_x} \Phi^\lambda(x, \frac{y}{\epsilon}) V_\lambda^{(s)}(x) \; d\sigma_x \\ &\quad - \int_{\partial D} \Phi^\lambda(x, \frac{y}{\epsilon}) \frac{\partial}{\partial n} V_\lambda^{(s)}(x) \; d\sigma_x \end{aligned}$$

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(3) $\lambda \to \infty$ may, e.g., be treated by a combination with appropriate "geometric optics" and stationary phase.

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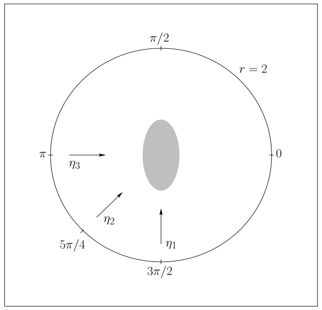

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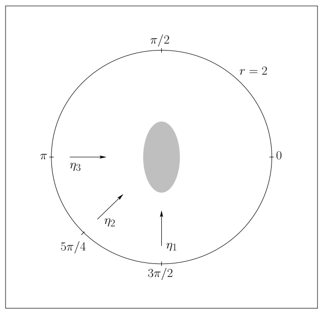

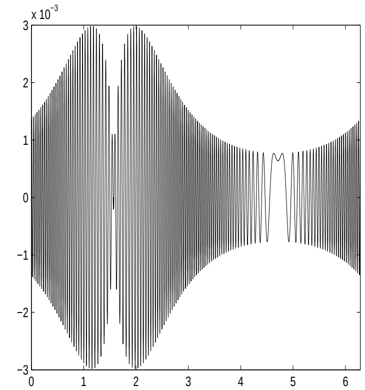

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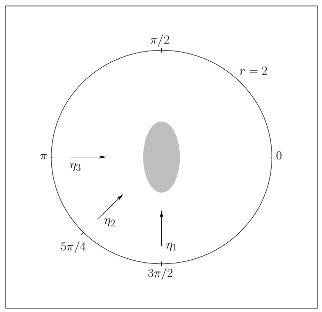

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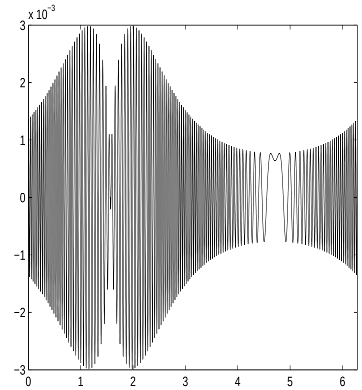

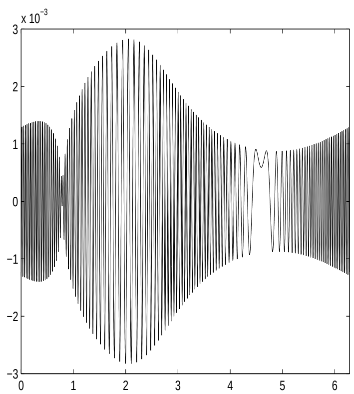

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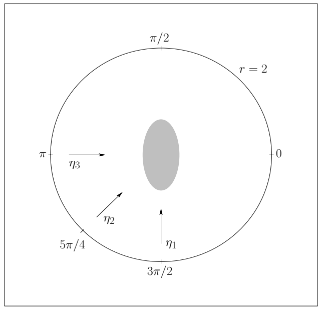

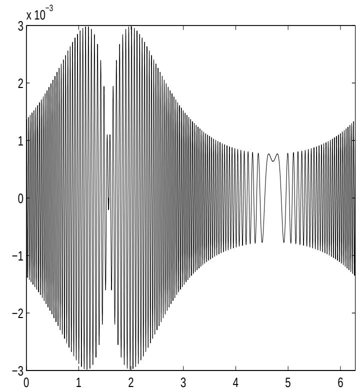

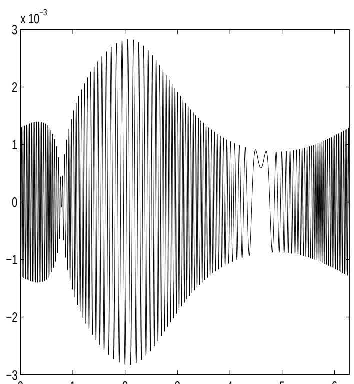

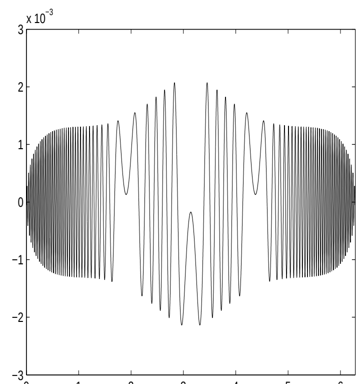

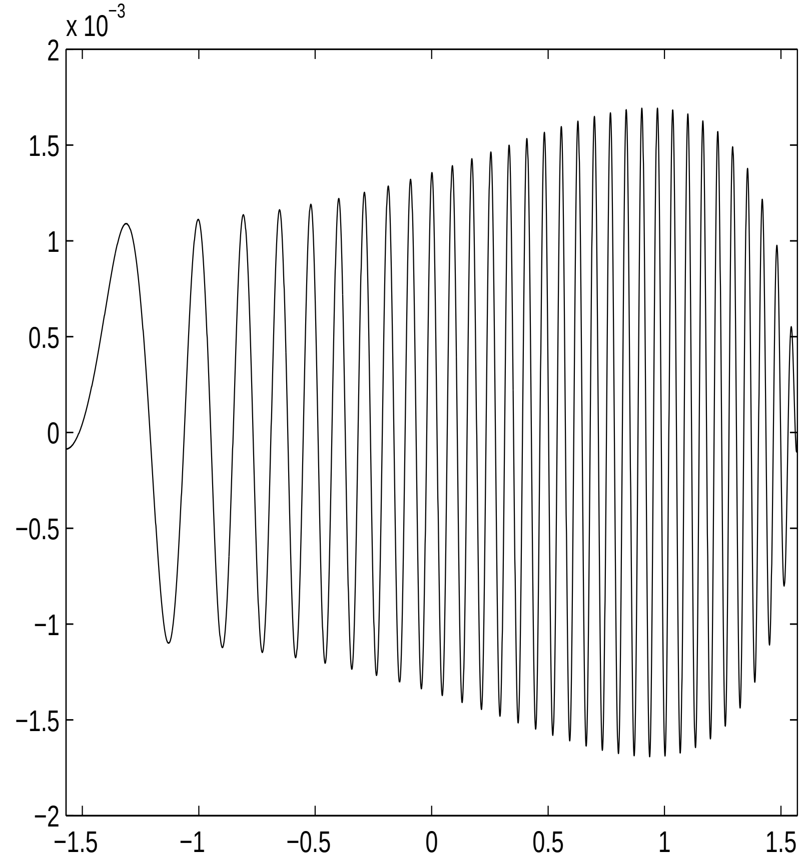
x 10^−3
2
1.5
1
0.5
0
−0.5
−1
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−2
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−1
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0
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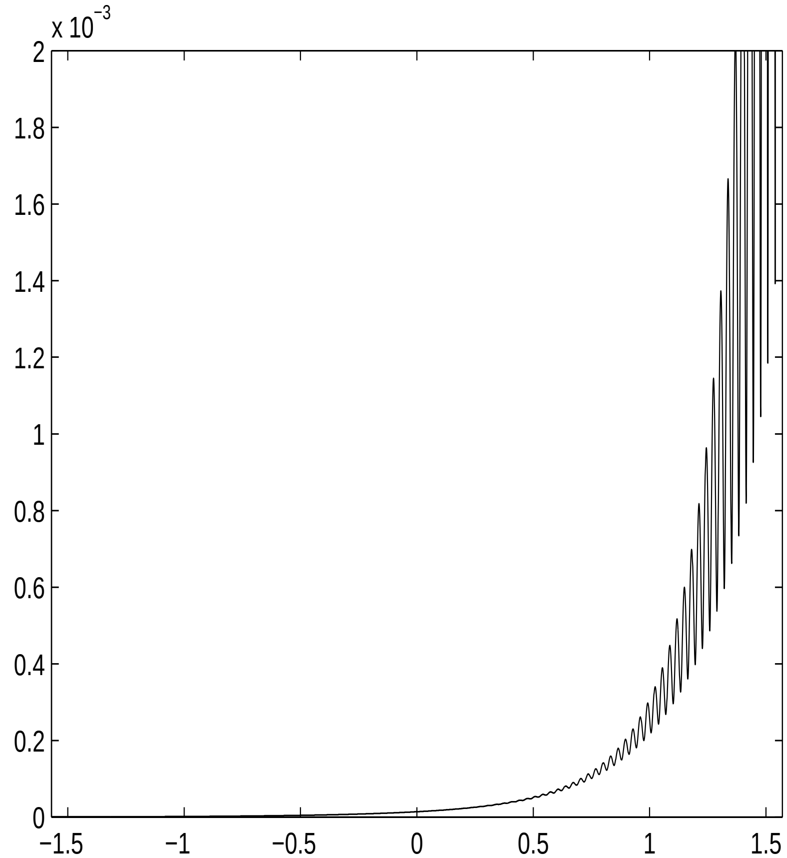
1
1.5


x 10^−3
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−1.5
−1
−0.5
0
0.5
1
1.5