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Jacobi's Ideas on Eigenvalue Computation in a modern context

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General remarks

$$Ax = \lambda x$$

Nonlinear problem:

for $n > 4$ no explicit solution

Essentially iterative methods

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No matrix notation in that time

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Masterthesis of Anjet de Boer, 1991, Utrecht

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**Sur les Variations séculaires des Eléments elliptiques
des sept Planètes principales: Mercure, Vénus, la Terre, Mars,
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linear eigensystem from system of 7 diff. equations

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**Perturbations to the orbits of planets caused by the
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linear eigensystem from system of 7 diff. equations

coefficients of characteristic polynomial

He neglected some small elements: factors of degree 3 and 4

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Papers by Jacobi (1804-1851)

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New method for solution of sym. linear systems;

Jacobi-rotations as "preconditioner" for G-Jacobi method

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He announces the application for eigenproblems

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Claim: easier and more accurate method (unsupported)

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Bodewig (1951): **Jacobi knew his methods before 1840**

(inconclusive) evidence: letter of Schumacher to Gauss (**1842**)

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Whittaker (1924) described G-Jacobi

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The Jacobi (rotation) method was forgotten, but J. described the two methods
as one single algorithm

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Jacobi method (2)

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Bodewig (1950, 1951) described the full J-method

He claimed the rediscovery

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(2) **Consider orthogonal complement of e_1 :**

$$A \begin{pmatrix} 1 \\ w \end{pmatrix} = \begin{pmatrix} a_{1,1} & c^T \\ c & F \end{pmatrix} \begin{pmatrix} 1 \\ w \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ w \end{pmatrix}$$

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leads to

$$\lambda = a_{1,1} + c^T w$$

$$(F - \lambda I)w = -c$$

Jacobi (2)

start with $w = 0, \theta = a_{1,1}$

Solve w from $(F - \theta I)w = -c$ with G-J iterations

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Goldstine suggested J's rotations only for proving real eigenvalues

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Krylov suggested in 1931 the subspace:

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for some convenient starting vector x

for construction of **characteristic polynomial**

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illconditioned basis, but in his case: $m = 6$

How to make things work for **large m** ?

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In the early 1950s: **orthogonal basis**

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Normalize: v_2 **(so far nothing new!)**

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Orthogonalize w.r.t v_1, v_2 and normalize: v_3

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Results in **well-conditioned basis** (Stewart, SIAM books)

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A unsymmetric: ARNOLDI METHOD (1952)

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Davidson opens ways for other subspaces

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Idea: apply preconditioner instead of Jacobi rotations and

use Jacobi's idea for new update of z

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$(A - \theta I)$ restricted to z^\perp is given by

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Expand subspace with (approx.) solution of $Bt = r$

Jacobi-Davidson method, SIMAX 1996

Newton method for RQ

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Numerical example

$$n = 100, A = \mathbf{tridiag}(1, 2.4, 1)$$

$$x = (1, 1, \dots, 1)^T$$

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Davidson, prec. with GMRES(5) for $(A - \theta_k I)\tilde{t} = r$:

slow convergence (since $\theta_k \approx \lambda$)

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Jac.Dav., **GMRES(5)** for $F\tilde{t} = r$ with

$$F = (I - zz^T)(A - \theta_k I)(I - zz^T): **13 it's**$$

Note that F has no small eigenvalues

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More practical example

Acoustics, attachment line:

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For problem coming from **acoustics**:

A, C 19-diagonal, B complex, $n = 136161$

Results for interior isolated eigenvalue (resonance)

on a **CRAY T3D**

Processors	Elapsed time (sec)
16	206.4
32	101.3
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For $n = 274625$, on 64 processors: **93.3 seconds**

1 invert step \approx 3 hours