Maximum-Norm Resolvent Estimates and Stability in Parabolic Finite Element Equations

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Dedicated to Michel Crouzeix

Parabolic problem

Initial-boundary value problem for heat equation
($\Omega \subset \mathbb{R}^2$, $\partial\Omega$ smooth),

$$u_t - \Delta u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0,$$
$$u(\cdot, 0) = v \quad \text{in } \Omega.$$

FEM: Triangulation $\mathcal{T}_h = \{\tau\}$ of Ω, $h = \max_{\mathcal{T}_h} \operatorname{diam}(\tau)$,

$$S_h = \{\chi \in \mathcal{C}(\overline{\Omega}) : \chi \text{ linear on each } \tau \in \mathcal{T}_h,\ \chi = 0 \quad \text{on } \partial\Omega\}.$$

Spatially semidiscrete problem, $(v, w) = \int_\Omega vw\, dx$:

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = 0,\ \forall \chi \in S_h,\ t \geq 0, \quad u_h(0) = v_h.$$

Discrete Laplacian: $\Delta_h : \; S_h \to S_h$, negative definite:

$$-(\Delta_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \psi, \chi \in S_h.$$

One may write the semidiscrete problem

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t \geq 0, \quad \text{with } u_h(0) = v_h.$$

Solution operator $u_h(t) = E_h(t) v_h = e^{\Delta_h t} v_h$.

With $\{\Lambda_j, \Phi_j\}$ eigensystem of $-\Delta_h$: $\quad u_h(t) = \sum_j e^{-\Lambda_j t} (v_h, \Phi_j)$.

Parseval's identity implies *Stability:*

$$\|u_h(t)\| = \|E_h(t) v_h\| \leq \|v_h\|, \quad \text{where } \|w\| = \|w\|_{L_2} = \Big(\int_\Omega |w|^2 dx \Big)^{1/2},$$

and *Smoothing property:*

$$\|\Delta_h u_h(t)\| = \|\Delta_h E_h(t) v_h\| = \|E_h'(t) v_h\| \leq C t^{-1} \|v_h\|, \quad \text{for } t > 0.$$

Spatially semidiscrete parabolic problem

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t \geq 0, \quad \text{with } u_h(0) = v_h.$$

Solution operator $u_h(t) = E_h(t)v_h = e^{\Delta_h t}$:

$$\|E_h(t)v_h\| + t\|E_h'(t)v_h\| \leq C\|v_h\|, \quad \text{for } t > 0.$$

Smooth and nonsmooth data error estimates: with v_h suitable:

$$\|u_h(t) - u(t)\| \leq \begin{cases} Ch^2\|v\|_{H^2}, & \text{if } v = 0 \quad \text{on } \partial\Omega, \\ \\ Ch^2 t^{-1}\|v\|. \end{cases}$$

Also uses elliptic error estimate $\|R_h v - v\| \leq Ch^2\|v\|_{H^2}$ where

Ritz projection $R_h : H_0^1 \to S_h$: $(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi),\ \forall \chi \in S_h.$

Maximum-norm estimates. Continuous problem

$u_t - \Delta u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t \geq 0, \quad u(\cdot, 0) = v \quad \text{in } \Omega.$

Solution operator $E(t)$: $u(t) = E(t)v = e^{\Delta t}v$.

Maximum-principle implies $\|E(t)v\|_{\mathcal{C}} \leq \|v\|_{\mathcal{C}} = \sup_{x\in\Omega} |v(x)|$.

Analytic semigroup on $\mathcal{C}_0(\bar{\Omega}) = \{v \in \mathcal{C}(\bar{\Omega}) : \ v = 0 \text{ on } \partial\Omega\}$:

$$\|(\lambda I + \Delta)^{-1}v\|_{\mathcal{C}} \leq \frac{C}{1+|\lambda|}\|v\|_{\mathcal{C}}, \quad \text{for } \lambda \notin \Sigma_\delta = \{\lambda : \ |\arg\lambda| < \delta\},$$

where $\delta \in (0, \frac{1}{2}\pi)$ is arbitrary (Stewart 1974).

Implies *Smoothing estimate:*

$$\|E'(t)v\|_{\mathcal{C}} \leq \frac{C}{t}\|v\|_{\mathcal{C}}, \quad \text{for } t > 0, \ v \in \mathcal{C}_0(\bar{\Omega}).$$

Semidiscrete parabolic problem

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t > 0, \quad \text{with } u_h(0) = v_h.$$

Solution operator $E_h(t)$: $u_h(t) = E_h(t)v_h = e^{\Delta_h t}$.

No maximum-principle, $E_h(t)$ not a contraction in $\|\cdot\|_{\mathcal{C}}$. Stability?

Theorem (Schatz, Thomée and Wahlbin -80)**.**

Assume $\mathcal{T}_h$ quasiuniform. Then, with $\ell_h = \max(1, \log(1/h))$,

$$\|E_h(t)v_h\|_{\mathcal{C}} \leq C\ell_h \|v_h\|_{\mathcal{C}}, \quad \textit{for } t \geq 0,$$

and

$$\|\Delta_h E_h(t)v_h\|_{\mathcal{C}} = \|E_h'(t)v_h\|_{\mathcal{C}} \leq Ct^{-1}\ell_h \|v_h\|_{\mathcal{C}}, \quad \textit{for } t > 0.$$

Theorem. *Assume $\mathcal{T}_h$ quasiuniform. Then*

$$\|E_h(t)v_h\|_{\mathcal{C}} \leq C\ell_h\|v_h\|_{\mathcal{C}}.$$

Sketch of proof: We want to show

$$|E_h(t)v_h(x)| \leq C\ell_h\|v_h\|_{\mathcal{C}}, \quad \forall x \in \Omega.$$

Discrete delta-function: $\delta_h^x \in S_h$,

$$(\delta_h^x, \chi) = \chi(x), \quad \forall \chi \in S_h.$$

Discrete fundamental solution: $\Gamma_h^x(t) = E_h(t)\delta_h^x$.

One notes $\quad E_h(t)v_h(x) = (\Gamma_h^x(t), v_h)$, so

$$|E_h(t)v_h(x)| \leq \|\Gamma_h^x(t)\|_{L_1}\|v_h\|_{\mathcal{C}}.$$

Thus show:

$$\|\Gamma_h^x(t)\|_{L_1} \leq C\ell_h.$$

Theorem. *Assume $\mathcal{T}_h$ quasiuniform. Then*

$$\|E_h(t)v_h\|_{\mathcal{C}} \le C\ell_h\|v_h\|_{\mathcal{C}}.$$

We need to show, for $\Gamma_h^x(t) = E_h(t)\delta_h^x$,

$$\|\Gamma_h^x(t)\|_{L_1} \le C\ell_h.$$

Modified distance function $\rho_h^x(y) = (|x-y|^2 + h^2)^{1/2}$.
Then, with $\Gamma = \Gamma_h^x(t), \rho = \rho_h^x$:

$$\|\Gamma(t)\|_{L_1} \le \|\rho^{-1}\|\,\|\rho\Gamma(t)\| \le C\ell_h^{1/2}\|\rho\Gamma(t)\|, \qquad \|\cdot\| = \|\cdot\|_{L_2}.$$

so we need

$$\|\rho\Gamma(t)\| \le C\ell_h^{1/2}.$$

Show $$\|\rho\Gamma(t)\| \le C\ell_h^{1/2}.$$

We have

$$(\Gamma_t, \chi) + (\nabla\Gamma, \nabla\chi) = 0, \quad \text{for } \chi \in S_h, \quad \text{with } \Gamma(0) = \delta_h^x.$$

Consider

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|\rho\Gamma\|^2 + \|\rho\nabla\Gamma\|^2 &= (\Gamma_t, \rho^2\Gamma) + (\nabla\Gamma, \nabla(\rho^2\Gamma)) - 2(\nabla\Gamma, \rho\,\nabla\rho\,\Gamma) \\ &= (\Gamma_t, \rho^2\Gamma - \chi) + (\nabla\Gamma, \nabla(\rho^2\Gamma - \chi)) - 2(\rho\nabla\Gamma, \nabla\rho\,\Gamma) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Choose $\chi = P_h(\rho^2\Gamma)$. Then $I_1 = 0$. Use an inverse estimate and superapproximation to obtain

$$|I_1| + |I_2| \le C(\|\Gamma\|^2 + \|\Gamma\|\,\|\rho\nabla\Gamma\|) \le \tfrac{1}{2}\|\rho\nabla\Gamma\|^2 + C\|\Gamma\|^2.$$

Show $$\|\rho\Gamma(t)\| \le C\ell_h^{1/2}.$$

By above,

$$\frac{d}{dt}\|\rho\Gamma\|^2 + \|\rho\nabla\Gamma\|^2 \le C\|\Gamma\|^2.$$

Hence

$$\|\rho\Gamma(t)\|^2 + \int_0^t \|\rho\nabla\Gamma\|^2\,ds \le \|\rho\delta_h^x\|^2 + C\int_0^t \|\Gamma\|^2\,ds.$$

Here $\|\rho\delta_h^x\| \le C$ and, by energy arguments,

$$\int_0^t \|\Gamma\|^2 ds \le \|(-\Delta_h)^{-1}\delta_h^x(x)\|^2 \le C\ell_h.$$

This shows the desired estimate for $\|\rho\Gamma(t)\|$.

Initial-value problem in Banach space $\mathcal{B}$.

$$u_t + Au = 0 \quad \text{for } t > 0, \quad \text{with } u(0) = v.$$

Solution operator $E(t) = e^{-At}$, semigroup.

Ex. $A = -\Delta$, $\mathcal{B} = \mathcal{C}_0(\Omega)$, or $A = -\Delta_h$, $\mathcal{B} = S_h$, with $\|\cdot\| = \|\cdot\|_{\mathcal{C}}$.

Theorem. *Assume $E(t) = e^{-At}$ semigroup in $\mathcal{B}$, $\|E(t)v\| \le C\|v\|$. Then the following conditions are equivalent:*

(i) With $K > 0$,

$$\|E(t)\| + t\|E'(t)\| \le K, \quad \textit{for } t \ge 0.$$

(ii) With $\delta \in (0, \frac{1}{2}\pi), M > 0$

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda|}, \quad \forall\ \lambda \notin \Sigma_\delta.$$

(iii) $E(t)$ is analytic in a sector around $\mathbb{R}_+$.

Theorem. *Assume, with* $\delta \in (0, \frac{1}{2}\pi), M > 0$,

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda|}, \quad \forall\ \lambda \notin \Sigma_\delta.$$

Then, with $\Gamma = \partial\Sigma_\beta,\ \beta \in (\delta, \frac{1}{2}\pi)$, *say,*

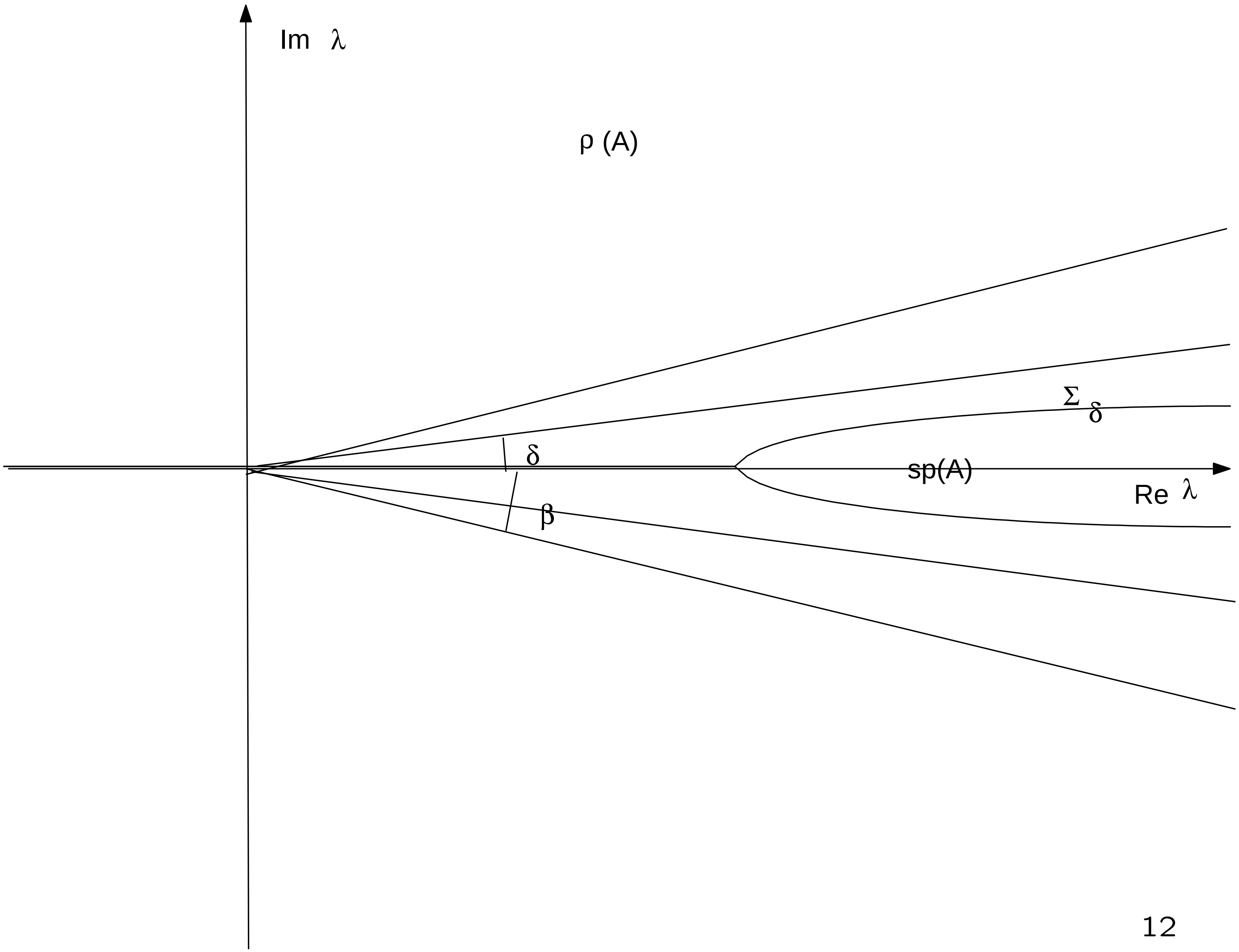
$$E(t) = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(\lambda I - A)^{-1}\, d\lambda$$

defines a bounded semigroup in $\mathcal{B}$, *and for some* $K > 0$,

$$\|E(t)\| + t\|E'(t)\| \le K \quad \text{for } t \ge 0.$$

$E(t)$ *is analytic in a sector around* $\mathbb{R}_+$.

Such a semigroup is called an analytic semigroup.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Homogeneous semidiscrete parabolic equation

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t \geq 0, \quad u_h(0) = v_h.$$

Solution $u_h(t) = E_h(t)v_h = e^{\Delta_h t}$. Recall:

Theorem. *Assume $\mathcal{T}_h$ quasiuniform. Then*

$$\|E_h(t)v_h\|_{\mathcal{C}} + t\|E_h'(t)v_h\|_{\mathcal{C}} \leq C\ell_h\|v_h\|_{\mathcal{C}}, \quad \textit{for } t > 0.$$

By semigroup theory this implies a resolvent estimate:

$$\|(\lambda I + \Delta_h)^{-1}v_h\|_{\mathcal{C}} \leq \frac{C\ell_h^2}{|\lambda|}\|v_h\|_{\mathcal{C}}, \quad \forall\, \lambda \notin \Sigma_{\delta_h}, \quad \text{where } \delta_h = \tfrac{1}{2}\pi - c\ell_h^{-2}.$$

This in turn implies *stability* and *smoothing estimates:*

$$\|E_h(t)v_h\|_{\mathcal{C}} \leq C\ell_h^2 \log \ell_h\|v_h\|_{\mathcal{C}} \quad \text{and} \quad t\|E_h'(t)v_h\|_{\mathcal{C}} \leq C\ell_h^4\|v_h\|_{\mathcal{C}}.$$

Weaker than the above!

Theorem (Schatz, Thomée and Wahlbin -98).
Assume $\mathcal{T}_h$ quasiuniform. Then

$$\|E_h(t)v_h\|_{\mathcal{C}} + t\|E_h'(t)v_h\|_{\mathcal{C}} \le C\|v_h\|_{\mathcal{C}}, \quad \text{for } t > 0. \qquad (*)$$

No factor ℓ_h!

By semigroup theory: There exists $\varphi \in (0, \frac{\pi}{2})$ such that

$$\|(\lambda I + \Delta_h)^{-1}v_h\|_{\mathcal{C}} \le \frac{C}{1+|\lambda|}\|v_h\|_{\mathcal{C}}, \quad \forall\ \lambda \notin \Sigma_\varphi.$$

Theorem (Bakaev, Thomée and Wahlbin -01).
Assume $\mathcal{T}_h$ quasiuniform. Then, for ANY $\varphi \in (0, \frac{\pi}{2})$,

$$\|(\lambda I + \Delta_h)^{-1}v_h\|_{\mathcal{C}} \le \frac{C}{1+|\lambda|}\|v_h\|_{\mathcal{C}}, \quad \forall\ \lambda \notin \Sigma_\varphi.$$

By semigroup theory: new proof of (*).

Nonquasiuniform triangulations $\mathcal{T}_h$.

Lumped Mass Method: Quadrature

$$(\psi,\chi)_h = \sum_{\tau\in\mathcal{T}_h} Q_{\tau,h}(\psi\,\chi), \quad \text{where } Q_{\tau,h}(f) = \tfrac{1}{3}\,\text{area}(\tau)\sum_{j=1}^{3} f(P_{\tau,j}) \approx \int_\tau f\,dx.$$

Modified semidiscrete problem

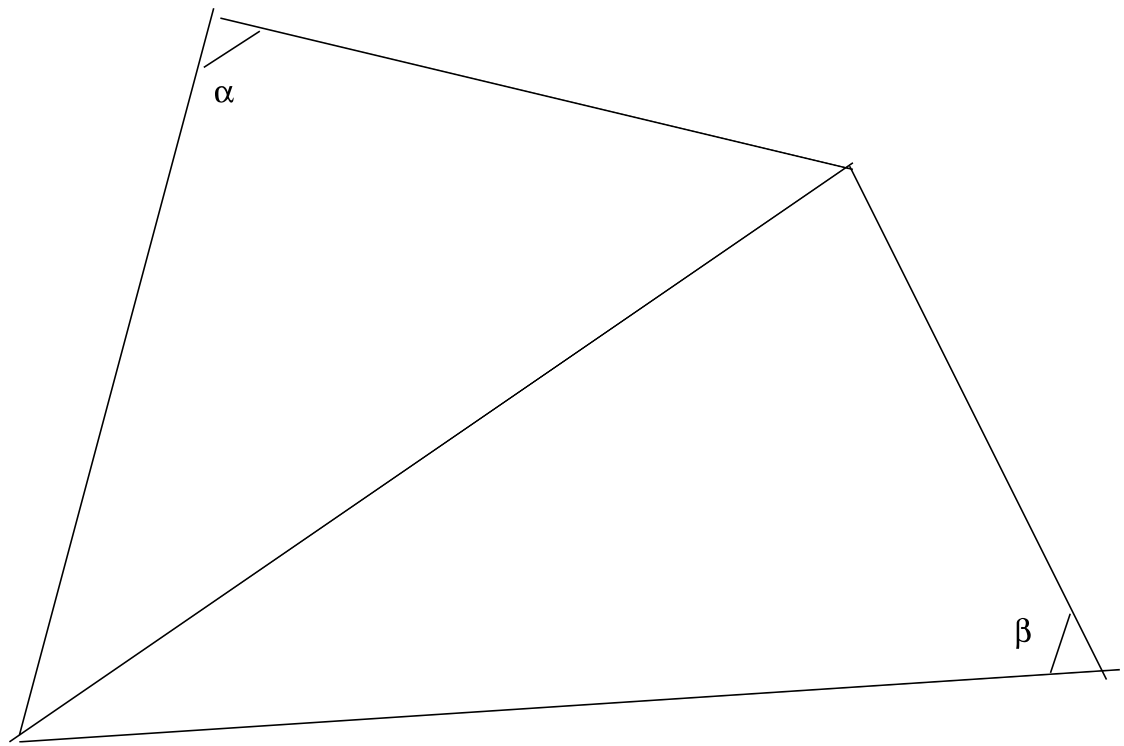
$$(u_{h,t},\chi)_h + (\nabla u_h,\nabla\chi) = 0, \text{ for } t\ge 0, \quad u_h(0) = v_h.$$

Solution operator $\overline{E}_h(t) = e^{\overline{\Delta}_h t}$ where

$$-(\overline{\Delta}_h\psi,\chi)_h = (\nabla\psi,\nabla\chi), \quad \forall\psi,\chi\in S_h.$$

Lumped mass: Replace mass matrix (m_{jk}) by diagonal matrix $(\bar{m}_{jk})$ with diagonal elements $\bar{m}_{jj} = \sum_k m_{jk}$.

Delaunay triangulation: $\alpha + \beta \leq \pi$



Theorem (Fujii -73). *Assume $\mathcal{T}_h$ is of Delaunay type. Then a maximum-principle holds and*

$$\|\overline{E}_h(t)v_h\|_{\mathcal{C}} \le \|v_h\|_{\mathcal{C}}, \quad \textit{for } t \ge 0.$$

This contraction semigroup is, in fact, an analytic semigroup.
Theorem (Crouzeix, Thomée -01). *We have*

$$\|(\lambda I + \overline{\Delta}_h)^{-1}v_h\|_{\mathcal{C}} \le \frac{C\ell_h^{1/2}}{1+|\lambda|}\|v_h\|_{\mathcal{C}}, \quad \lambda \notin \Sigma_{\delta_h}, \quad \delta_h = \tfrac{\pi}{2} - c\ell_h^{1/2}.$$

For proof, first use energy arguments to show the corresponding estimate in discrete L_p norm, for any $p < \infty$.

It follows that

$$\|\overline{E}_h'(t)v_h\|_{\mathcal{C}} \le C\ell_h t^{-1}\|v_h\|_{\mathcal{C}}, \quad \text{for } t > 0.$$

Nonquasiuniform triangulations $\mathcal{T}_h$. Standard FEM.

$P_h: L_2 \to S_h$ L_2-projection onto S_h: $(P_h v, \chi) = (v, \chi), \quad \forall \chi \in S_h.$

Crouzeix, Thomée -87: For $\tau_0 \in \mathcal{T}_h$, let $Q_j(\tau_0)$ denote the set of triangles which are "j triangles away from τ_0".

Lemma. *Assume that* $supp(v) \subset \tau_0$. *Then*

$$\|P_h v\|_{L_2(\tau)} \le C\gamma^j \|v\|_{L_2}, \quad \textit{for } \tau \in Q_j(\tau_0), \quad \textit{where } \gamma = 0.318.$$

Let $n_j(\tau_0) = \#$ triangles in $Q_j(\tau_0)$.
Assume, for some $\alpha \ge 1, \beta \ge 1$ (if $\alpha > 1$ we can choose $\beta = \alpha^4$),

$$h_\tau / h_{\tau_0} \le C\alpha^j, \quad n_j(\tau) \le C\beta^j, \quad \text{for } \tau \in Q_j(\tau_0),\ \forall \tau_0 \in \mathcal{T}_h.$$

Theorem. *If* $\alpha\beta\gamma < 1$, *then* $\|P_h v\|_{\mathcal{C}} \le C\|v\|_{\mathcal{C}}$.

Lemma. *Assume that* $supp(v) \subset \tau_0$*. Then*

$$\|P_h v\|_{L_2(\tau)} \leq C\gamma^j \|v\|_{L_2}, \quad \textit{for } \tau \in Q_j(\tau_0), \quad \textit{where } \gamma = 0.318.$$

Assume, for $\alpha \geq 1, \ \beta \geq 1$,

$$h_\tau / h_{\tau_0} \leq C\alpha^j, \quad n_j(\tau) \leq C\beta^j, \quad \text{for } \tau \in Q_j(\tau_0), \ \forall \tau_0 \in \mathcal{T}_h.$$

Theorem (Bakaev, Crouzeix, Thomée -06)**.**
If $\alpha^2\beta\gamma < 1$*, then, for any* $\delta \in (0, \frac{\pi}{2})$*,*

$$\|(\lambda I + \Delta_h)^{-1} v_h\|_{\mathcal{C}} \leq \frac{C\ell_h^{1/2}}{1 + |\lambda|} \|v_h\|_{\mathcal{C}}, \quad \forall \lambda \notin \Sigma_\delta.$$

With $\gamma = 0.318, \ \beta = \alpha^4$, $\alpha^2\beta\gamma < 1$ holds if $\alpha < 1.21$.

This permits seriously nonquasiuniform $\mathcal{T}_h$. Proof by energy arguments similar to earlier, and uses the exponential decay lemma.

Time stepping, fully discrete schemes

Spatially semidiscre problem. Find $u_h(t) \in S_h$:

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = 0, \quad \forall \chi \in S_h,\ t \geq 0, \quad \text{with } u_h(0) = v_h.$$

Let $k =$ time step, $t_n = nk,\ U^n \approx u(t_n),\ \overline{\partial} U^n = (U^n - U^{n-1})/k$.

Backward Euler: Find $U^n \in S_h$ with $U^0 = v_h$:

$$(\overline{\partial} U^n, \chi) + (\nabla U^n, \nabla \chi) = 0, \quad \forall \chi \in S_h,\ n \geq 1.$$

Crank-Nicolson: with $\overline{U}^n = (U^n + U^{n-1})/2$,

$$(\overline{\partial} U^n, \chi) + (\nabla \overline{U}^n, \nabla \chi) = 0, \quad \forall \chi \in S_h,\ n \geq 1.$$

These may be written, with $E_{kh} = r(-k\Delta_h)$,

$$U^n = E_{kh} U^{n-1} = E_{kh}^n v_h, \quad r(\lambda) = \begin{cases} 1/(1+\lambda), \\ (1-\lambda/2)/(1+\lambda/2). \end{cases}$$

Time stepping in Banach space $\mathcal{B}$.

Assume $-A$ generates analytic semigroup in $\mathcal{B}$,

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{1+|\lambda|} \quad \text{for } \lambda \notin \Sigma_\delta, \quad \delta \in (0, \tfrac{1}{2}\pi).$$

Then, for $R(\lambda)$ rational function, bounded in Σ_θ, $\theta \in (\delta, \frac{1}{2}\pi]$,

$$R(A) = R(\infty)I + \frac{1}{2\pi i}\int_\Gamma R(\lambda)(\lambda I - A)^{-1}\, d\lambda, \quad \Gamma \text{ suitable.}$$

Set $\quad U^n = E_k^n v, \quad$ where $\ E_k = r(kA). \quad$ Stability?

Theorem (Crouzeix, Larsson, Piskarev, Th. -93, Palencia -93)**.** *Assume $r(\lambda)$ is $A(\theta)-$stable, $\theta \in (\delta, \frac{1}{2}\pi]$. Then*

$$\|U^n\| = \|E_k^n v\| \le CM\|v\|, \quad \textit{for } t_n \ge 0.$$

Theorem. *If* $r(\lambda)$ $A(\theta)-$*stable,* $\theta \in (\delta, \frac{1}{2}\pi]$, *then* $\|E_k^n v\| \le CM\|v\|$.

Proof uses, with suitable $\Gamma = \Gamma_n$,

$$r(A)^n = r(\infty)^n I + \frac{1}{2\pi i}\int_\Gamma r(\lambda)^n(\lambda I - A)^{-1}\,d\lambda.$$

For $|r(\infty)| = 1$ one needs to study $r(\lambda)$ near $\lambda = \infty$. For CN:

$$|r(\lambda)| = \Big|\frac{1-\lambda/2}{1+\lambda/2}\Big| = e^{-4/\lambda + O(1/\lambda^2)}, \quad \text{for } |\lambda| \text{ large.}$$

Example for fully discrete scheme:

Assume $\mathcal{T}_h$ nonquasiuniform as above and use CN for time-stepping. Then

$$\|E_{kh}^n v_h\|_{\mathcal{C}} \le C\ell_h^{1/2}\|v_h\|_{\mathcal{C}}, \quad \text{for } t_n \ge 0.$$

Summary

Cont. pr.: $E(t) = e^{\Delta t}$: $\|E(t)\|_{\mathcal{C}} \le 1,\ \|E'(t)\|_{\mathcal{C}} \le Ct^{-1}$. (S -74)

S_h piecewise linears, $\mathcal{T}_h$ quasiuniform, $E_h(t) = e^{\Delta_h t}$:

$$\|E_h(t)\|_{\mathcal{C}} + t\|E'(t)\|_{\mathcal{C}} \le C\ell_h. \qquad \text{(STW -80)}$$

This implies

$$\|(\lambda I + \Delta_h)^{-1}\| \le \frac{C\ell_h^2}{1+|\lambda|}, \quad \forall\ \lambda \notin \Sigma_{\delta_h},\ \delta_h = \tfrac{\pi}{2} - c\ell_h^{-2}.$$

Sharper result: $\|E_h(t)\|_{\mathcal{C}} + t\|E'(t)\|_{\mathcal{C}} \le C$. (STW -98)
Hence

$$\|(\lambda I + \Delta_h)^{-1}\| \le \frac{C}{1+|\lambda|}, \quad \forall\ \lambda \notin \Sigma_{\delta},\ \text{some } \delta \in (0, \tfrac{\pi}{2}).$$

Holds for all $\delta \in (0, \frac{\pi}{2})$. (BTW -01).

Nonquasiuniform $\mathcal{T}_h$:

$\mathcal{T}_h$ of Delaunay type: $\overline{E}_h(t) = e^{\overline{\Delta}_h t}$: Max-principle $\|\overline{E}_h(t)\|_{\mathcal{C}} \le 1$.

$$\|(\lambda I + \overline{\Delta}_h)^{-1}\|_{\mathcal{C}} \le \frac{C\ell_h}{1+|\lambda|}, \quad \lambda \notin \Sigma_{\delta_h},\ \delta_h = \tfrac{\pi}{2} - c\ell_h^{1/2}. \quad \text{(CT -01)}$$

Standard FEM

Assume $h_\tau/h_{\tau_0} \le C\alpha^j$ if τ is j triangles away from τ_0, $\alpha = 1.2$,

$$\|(\lambda I + \Delta_h)^{-1}\|_{\mathcal{C}} \le \frac{C\ell_h^{1/2}}{1+|\lambda|}, \quad \forall\ \lambda \notin \Sigma_\delta,\ \delta \in (0, \tfrac{\pi}{2}). \quad \text{(BCT -06)}$$

Fully discrete schemes

$\mathcal{T}_h$ as above, $E_{kh} = r(-k\Delta_h)$, $r(\lambda)$ $A(\theta)$-stable, $\theta \in (0, \frac{\pi}{2})$. Then

$$\|E_{kh}^n v_h\|_{\mathcal{C}} \le C\ell_h^{1/2}\|v_h\|_{\mathcal{C}}, \quad \text{for } t_n \ge 0. \quad \text{(CLPT -93)}$$