Dissipative hyperbolic systems:

asymptotic behavior and numerical approximation

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Hyperbolic systems of balance laws

$$u_t + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = G(u), \; u \in \Omega \subseteq \mathbb{R}^n \tag{1}$$

$$u(x,0) = u_0(x) \tag{2}$$

Main topic of the Talk:

Asymptotic behavior of smooth solutions for large times

- **Rate of decay to a constant equilibrium state**
- **Asymptotic convergence to the solutions to a suitable profile (linearized systems or Chapman-Enskog expansion)**
- **Consistent numerical approximations (increasingly accurate for long times)**

p-system with relaxation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \\ \partial_t v + \partial_x \sigma(u) = h(u) - v \qquad \sigma'(u) > |h'(u)| \end{cases}$$

$h(u) \equiv 0$ **damping;** $\sigma(u) = \lambda^2 u$ **Jin-Xin relaxation**

Nishida1978 , Hsiao-Liu 1992, Jin-Xin 1995, N. 1996....

Euler equations with damping

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) + \frac{1}{\gamma}\nabla \rho^{\gamma} = -v \end{cases}$$

Sideris, Thomases & Wang 2003, Coulombel & Goudon 2004

Part I: Backgrounds

The Basic Relaxation Structure

k **conserved quantities**

$$\begin{cases} \partial_t u_1 + \sum_j \partial_{x_j} F_1^j(u) = 0 \\ \\ \partial_t u_2 + \sum_j \partial_{x_j} F_2^j(u) = q(u) \end{cases}$$

with $u = (u_1, u_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$

The entropy structure

Chen-Liu-Levermore (1994); Boillat-Ruggeri (1998).

(E) $\exists$ **a strictly convex function** $\mathcal{E} = \mathcal{E}(u)$ **and some related entropy-flux** $\mathcal{F}_j = \mathcal{F}_j(u)$**, s.t. (for smooth solutions):**

$$\partial_t \mathcal{E}(u) + \sum_j \partial_{x_j} \mathcal{F}_j(u) = \mathcal{E}' \cdot G \qquad (3)$$

where $\mathcal{F}_j{}' = {}^T F_j' \mathcal{E}'$

Definition1 **The constant value** $\overline{u}$ **is an** ***equilibrium point*** **for system (1) if**

$$G(\overline{u}) = 0$$

Definition 2 **System (1) is** ***entropy dissipative*****, if, for every equilibrium point** $\overline{u}$ **and for all** $u \in \Omega$**,**

$$(\mathcal{E}'(u) - \mathcal{E}'(\overline{u})) \cdot G(u) \leq 0$$

The "Entropy" symmetric form

Set $U = (U_1, U_2) = \mathcal{E}'(u)$

$\mathcal{E}$ **strictly convex** $\Rightarrow \mathcal{E}'$ **invertible** $\Rightarrow \Phi(U) := (\mathcal{E}')^{-1}(U)$

$A_0(U) := \Phi'(U)$, symm, positive def., $C_j(U) := F_j'(\Phi(U))\Phi'(U)$, symm

$$A_0(U)\partial_t U + \sum_j C_j(U)\partial_{x_j} U = -\begin{pmatrix} 0 \\ D(U)U_2 \end{pmatrix}$$

with $D = D(U)$ **positive def.** $\Leftarrow$ **Assumption: Strictly entropy dissipative**

Remark

Entropy dissipation is too weak to prove global existence of smooth solutions

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0$$
$$\partial_t v + \partial_x v = -v$$

with the trivial dissipative entropy $\mathcal{E} = \frac{1}{2}(u^2 + v^2)$.

NEED FOR COUPLING CONDITIONS

The Shizuta-Kawashima condition

Consider our original system

$$\partial_t u + \sum_j F_j'(u)\partial_{x_j} u = G(u)$$

CONDITION SK (***Shizuta-Kawashima (1985)***)

Any eigenvector of $\sum_j F_j'(0)\xi_j$ is not in the null space of $G'(0)$ for every $\xi \in \mathbb{R}^m \setminus \{0\}$

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CONDITION SK (FOR THE ENTROPY FRAMEWORK)

For every $U \neq 0$ and $\lambda \in \mathbb{R}$

$$[\lambda A_0(0) + \sum_j C'_j(0)\xi_j] \begin{pmatrix} U \\ 0 \end{pmatrix} \neq 0 \qquad (K)$$

A global existence result

Theorem 1 ***(B. Hanouzet-R. N. ARMA 2003 in 1D, W.A. Yong 2004 in MultiD)*****. Assume that system (1) is strictly entropy dissipative and condition (SK) is satisfied. Then there exists $\delta > 0$ such that, if $\|u_0\|_s \leq \delta$, with $s \geq [m/2] + 2$, there is a unique global solution u of (1)–(2), which verifies**

$$u \in C^0([0,\infty); H^s(\mathbb{R}^m)) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}^m)),$$

and such that, in terms of the entropy variable U,

$$\sup_{0 \leq t < +\infty} \|U(t)\|_s^2 + \int_0^{+\infty} \left(\|\nabla U_1(\tau)\|_{s-1}^2 + \|U_2(\tau)\|_s^2\right) d\tau \leq C(\delta)\|U_0\|_s^2,$$

where $C(\delta)$ is a positive constant.

Some references about global existence

- T. Nishida (1978): p-system with damping
- T. T. Li (1994): Book (totally dissipative systems or diagonal structure)
- L. Hsiao and D. Serre (1996): p-system with variable entropy
- Y. Zeng (1999): Gas dynamics in thermal nonequilibrium (no (SK) condition!!)
- T. Sideris, B. Thomases, and D. Wang (2003): 3D Euler equations with damping

PART II: THE ASYMPTOTIC BEHAVIOR

S. Bianchini, B. Hanouzet, R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, to appear in Communications in Pure and Applied Mathematics

`http://www.iac.rm.cnr.it/~natalini/ps/BRSprint.pdf`

The linearized problem

$$\partial_t w + \sum_j A_j \partial_{x_j} w = Bw \tag{4}$$

$$B = \begin{bmatrix} 0 & 0 \\ D_1 & D_2 \end{bmatrix}, \qquad D_1 \in \mathbb{R}^{k\times(n-k)} D_2 \in \mathbb{R}^{(n-k)\times(n-k)} \tag{5}$$

- **(H1: Strictly entropy dissipative) $\exists\, A_0$ symmetric positive s.t. $A_j A_0$ is symmetric**

$$BA_0 = -\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}, \; D \in \mathbb{R}^{(n-k)\times(n-k)} \text{ positive definite}$$

- **(H2: (SK)) any eigenvector of $\sum_j A_j \xi_j$ is not in the null space of B**

A useful example: the wave equation with damping

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \\ \partial_t v + \partial_x u = -v \end{cases}$$

Main features:

- **symmetric system**
- **the source term is simple (only v)**
- **energy equality**

$$\frac{1}{2}\int |u|^2 + |v|^2 dx + \int_0^t \int |v|^2 dx dt = cost.$$

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All dissipative systems have this structure !!

The Conservative–Dissipative decomposition

$Q_0 = R_0 L_0$ **Null projector on the null space of** B

$Q_- = I - Q_0 = R_- L_-$ **complementary projector**

Set

$$w_c = L_0 A_0^{-1/2} w = \left[\begin{array}{cc} (A_{0,11})^{-1/2} & 0 \end{array} \right] w$$

$$w_d = L_- A_0^{-1/2} w = \left[\begin{array}{cc} 0 & ((A_0^{-1})_{22})^{-1/2} \end{array} \right] A_0^{-1} w$$

The CONSERVATIVE-DISSIPATIVE form

$$\partial_t \begin{pmatrix} w_c \\ w_d \end{pmatrix} + \tilde{A} \partial_x \begin{pmatrix} w_c \\ w_d \end{pmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D} \end{bmatrix} \begin{pmatrix} w_c \\ w_d \end{pmatrix} \tag{6}$$

where $\tilde{A}$ **is symmetric and** $\tilde{D}$ **is strictly positive**

$$\tilde{D} \doteq ((A_0^{-1})_{22})^{-1} D ((A_0^{-1})_{22})^{-1}$$

Example: the p-system with relaxation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x \sigma(u) = h(u) - v \end{cases}$$

Set $\lambda = \sqrt{\sigma'(0)}$ **and** $a = h'(0)$**, and** $\nu := \lambda^2 - a^2 > 0$

$$A = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ a & -1 \end{pmatrix} \Rightarrow A_0 = \begin{pmatrix} 1 & a \\ a & \lambda^2 \end{pmatrix}$$

The linearized Conservative-Dissipative form: $w_c = u$, $w_d = \nu^{-\frac{1}{2}}(v - au)$

$$\begin{cases} \partial_t w_c + a\partial_x w_c + \nu^{\frac{1}{2}} \partial_x w_d = 0 \\ \partial_t w_d + \nu^{\frac{1}{2}} \partial_x w_c - a w_d = -w_d \end{cases}$$

Example: the p-system with relaxation

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The Conservative-Dissipative form: $w_c = u$**,** $w_d = \nu^{-\frac{1}{2}}(v - au)$

$$\begin{cases} \partial_t w_c + \partial_x (a w_c + \nu^{\frac{1}{2}} w_d) = 0 \\ \partial_t w_d + \partial_x (\nu^{-\frac{1}{2}}(\sigma(w_c) - a^2 w_c) - a w_d) = \nu^{-\frac{1}{2}}(h(w_c) - a w_c) - w_d \end{cases}$$

Behavior of the Green kernel in 1D

Let A given by

$$A = \begin{bmatrix} A_{00}(t,x) & A_{0-}(t,x) \\ A_{-0}(t,x) & A_{--}(t,x) \end{bmatrix}$$

The 1D Green kernel $\Gamma(t,x) = \begin{bmatrix} \Gamma_{00}(t,x) & \Gamma_{0-}(t,x) \\ \Gamma_{-0}(t,x) & \Gamma_{--}(t,x) \end{bmatrix}$ **satisfies**

$$\begin{cases} \Gamma_t + A\Gamma_x & = & B\Gamma \\ \Gamma(0,x) & = & \delta(x)I \end{cases} \tag{7}$$

in Fourier transform

$$\hat{\Gamma}(t,\xi) = e^{E(i\xi)t}, \text{ where } E(z) = B - zA$$

The matrix $E(z)$ is in general not diagonalizable since B is negative definite but not symmetric ($E(z)$ is *permanently degenerate*), and out of exceptional points it is represented as

$$E(z) = \sum_j \lambda_j(z) P_j(z) + \sum_j D_j(z) \rightsquigarrow D \text{ nilpotent}$$

MAIN TOOL**: analysis of the behavior of the eigenvalues near the singular points**

Near $z = 0$: diffusive behavior

$$\lambda_{jk}(z) = -z\lambda_j^1 - z^2 c_{jk} + o(z^3). \tag{8}$$

$$g_t + \lambda_j^1 g_x = -\big(c_{jk} I + d_{jk}\big) g_{xx}, \qquad g \in \mathbb{R}^{m_{jk}}, \tag{9}$$

$c_{jk} = -\mu_{jk} - i\nu_{jk}, \qquad \mu_{jk} > 0,\ d_{jk}$ **positive definite**

$$g_{jk}(t,x) \doteq \underbrace{\frac{1}{2\gamma_{jk}\sqrt{\pi t}} \exp\left\{ -\frac{(x-\lambda_j^1 t)^2}{4(\mu_{jk}+i\nu_{jk})t} \right\}}_{\text{HEAT KERNEL}} \left[\sum_{\iota} M_{jk,\iota} \frac{(x-\lambda_j^1 t)^{2\iota}}{((\mu_{jk}+i\nu_{jk})t)^{\iota}} \right]$$

Near $z \sim \infty$: hyperbolic behavior

$$\lambda_{jk}(z) = -z\lambda_j - b_{jk} + \mathcal{O}(1/z), \tag{10}$$

$$h_t + \lambda_j h_x = (b_{jk} I + d_{jk})h, \tag{11}$$

$\Re e(b_{jk}) \le c < 0,\ d_{jk}$ **positive definite**

$$h_{jk}(t,x) = \underbrace{\delta(x - \lambda_j t)e^{b_{jk}t}}_{\text{DISSIPATIVE TRANSPORT}} \sum_{\iota} \frac{t^{\iota}}{\iota!}(\tilde{d}_{jk})^{\iota}$$

In between $0 < c \le |z| \le C$: (SK) condition

$$\Re(\lambda(i\xi)) \le -c\frac{|\xi|^2}{1+|\xi|^2} \Longrightarrow \text{ decay like } e^{-ct}$$

Define the matrix valued function $K(t,x) = \sum_{jk} K_{jk}(t,x)$

$$K_{jk} = \begin{bmatrix} r_j g_{jk}(t,x) p_{jk} l_j & -\overbrace{\partial_x}^{x\text{-derivative}} r_j g_{jk}(t,x) p_{jk} l_j A_{12} D^{-1} \\ -\underbrace{\partial_x}_{x\text{-derivative}} D^{-1} A_{21} r_j g_{jk}(t,x) p_{jk} l_j & \underbrace{\partial_{xx}}_{\text{two } x\text{-derivatives}} D^{-1} A_{21} r_j g_{jk}(t,x) p_{jk} l_j A_{21} D^{-1} \end{bmatrix}$$

and

$$\mathcal{K}(t,x) = \sum_{jk} R_j (h_{jk}(t,x) q_{jk}) L_j .$$

Theorem 2 *The Green kernel for (6) is*

$$\Gamma(t,x) = \underbrace{K(t,x)\chi\{\underline{\lambda}t \le x \le \bar{\lambda}t, t \ge 1\}}_{\text{diffusive}} + \underbrace{\mathcal{K}(t,x)}_{\text{dissipative hyperbolic}} + \overbrace{R(t,x)\chi\{\underline{\lambda}t \le x \le \bar{\lambda}t\}}^{\text{rest}}$$

where $\underline{\lambda}$, $\bar{\lambda}$ are the minimal and maximal eigenvalue of A and the rest $R(t,x)$ can be written as

$$R(t,x) = \sum_j \frac{e^{-(x-\lambda_j^1 t)^2/ct}}{1+t} \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(1)(1+t)^{-1/2} \\ \mathcal{O}(1)(1+t)^{-1/2} & \mathcal{O}(1)(1+t)^{-1} \end{bmatrix}$$

The Nonlinear case: decay to equilibrium

Consider now the original problem

$$w_t + F(w)_x = G(w), \qquad w(x,0) = w_0 \tag{12}$$

We have

$$w_t + F'(0)w_x - G'(0)w = \big(F'(0)w - F(w)\big)_x - \big(G'(0)w - G(w)\big)$$

Then we can write the solution as

$$w = \Gamma(t) * w_0 + \int_0^t \Gamma(t-\tau) * \big(F'(0)w - F(w)\big)_x d\tau - \int_0^t \Gamma(t-\tau) * \big(G'(0)w - G(w)\big) d\tau$$

Algebraic structure for the dissipative part of Γ

For every vector $(0, V) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ **one has**

$$K(t,x)\begin{pmatrix} 0 \\ V \end{pmatrix} = \sum_{jk} \boxed{\partial_x} \begin{pmatrix} -r_j g_{jk}(t,x) p_{jk} l_j A_{12} D^{-1} V \\ \partial_x D^{-1} A_{21} r_j g_{jk}(t,x) p_{jk} l_j A_{21} D^{-1} V \end{pmatrix}$$

x**-derivative!**

The rate of decay (in the general MultiD case)

Theorem 3 **Let $u(t)$ be the solution to the entropy strictly dissipative system (1) and take the Conservative-Dissipative variables $(w_c(t), w_d(t)) = Mu(t)$. Then, if $\|u(0)\|_{H^s}$ is bounded for s sufficiently large, the following decay estimates holds: for all multi index β, with $|\beta| = \bar{b}$,**

$$\|D^\beta w_c(t)\|_{L^p} \leq C \min\Big\{1, t^{-m/2(1-1/p)-\bar{b}/2}\Big\} \max\Big\{\|u(0)\|_{L^1}, \|u\|_{H^{\bar{b}+[m/2]+1}}\Big\}$$

$$\|D^\beta w_d(t)\|_{L^p} \leq C \min\Big\{1, t^{-m/2(1-1/p)-1/2-\bar{b}/2}\Big\} \max\Big\{\|u(0)\|_{L^1}, \|u\|_{H^{\bar{b}+[m/2]+1}}\Big\}$$

with $p \in [2, +\infty]$ if $m \geq 2$ and $p \in [1, +\infty]$ if $m = 1$

Some references about asymptotic behavior for dissipative hyperbolic systems

STABILITY OF CONSTANT STATES

- **T. Ruggeri and D. Serre (2002): general systems (zero mass of perturbation)**

ASYMPTOTIC DIFFUSIVE BEHAVIOR

- **L. Hsiao and T.-P. Liu (1992): p-system with damping**
- **I. Chern (1995): general 2x2 systems**
- **H. Liu and R.N. (2001): Jin-Xin system**

STABILITY OF SHOCK PROFILES

- **C. Mascia and K. Zumbrun (2002)**

Faster decay to the linearized system for $(m \geq 2)$

Assume that the system is already in the Conservative-Dissipative decomposition. Let $u^l = (u^l_c, u^l_d)$ be the solution to the linearized system

$$u^l_t + \sum_j^m f'_j(0) u^l_{x_j} = \begin{pmatrix} 0 \\ D_{u_d} q(0) u^l_d \end{pmatrix}$$

Theorem 4 **Under the assumptions of Theorem 3, for $m \geq 2$ and $p \in [2, \infty]$, we have the following decay estimate**

$$\|D^\beta(u(t) - u^l(t))\|_{L^p} \leq C \min\left\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2-1/2}\right\} E_{|\beta|+[m/2]+1}$$

Diffusive limit: the model case

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u = -v \end{cases}$$

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This is equivalent to

$$\overbrace{\partial_{tt} u - \partial_{xx} u + \partial_t u = 0}^{\text{dissipative wave equation}}$$

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But $\partial_t v$ **has a faster decay and** $v \sim \partial_x u$

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But $\partial_t v$ **has a faster decay and** $v \sim \partial_x u$

$$\Rightarrow \quad \overbrace{-\partial_{xx} u + \partial_t u = 0}^{\text{heat equation}}$$

Faster decay to the linear Chapman-Enskog expansion for $(m \geq 2)$

Assume that the system is already in the Conservative-Dissipative decomposition.

$$u_{c,t} + \sum_j A_{j,11}(0) u_{c,x_j} + \sum_j A_{j,12}(0) u_{d,x_j} = L_0 \sum_j (A_j(0)u - f_j(u))_{x_j}$$

$$\begin{aligned} u_{d,t} + \textstyle\sum_j A_{j,21}(0) u_{c,x_j} \quad & + \textstyle\sum_j A_{j,22}(0) u_{d,x_j} = D_{u_d} q(0) u_d \\ & + L_- \textstyle\sum_j (A_j(0)u - f_j(u))_{x_j} + \big(q(u) - D_{u_d} q(0) u_d\big) \end{aligned}$$

This yields (computing u_d by the second equation):

terms in black=associated linear parabolic part

$$u_{c,t} + \textstyle\sum_j A_{j,11}(0) u_{c,x_j} + \textstyle\sum_{j,k} A_{j,12}(0)(D_{u_d} q(0))^{-1} A_{k,21}(0) u_{c,x_j x_k}$$

$$= L_0 \textstyle\sum_j (A_j(0)u - f_j(u))_{x_j} + \textstyle\sum_j A_{j,12}(0)(D_{u_d} q(0))^{-1} \big(q(u) - D_{u_d} q(0) u_d\big)_{x_j}$$

$$- \textstyle\sum_j A_{j,12}(0)(D_{u_d} q(0))^{-1} \Big(u_{d,t x_j} + \textstyle\sum_k A_{k,22}(0) u_{d,x_j x_k} - \textstyle\sum_k L_- (A_k(0)u - f_k(u))_{x_j x_k}\Big)$$

terms in red=faster decay

Theorem 5 Let u_p be the solution of the associated linear parabolic problem. Under the assumptions of Theorem 3, for $m \geq 2$ and $p \in [2, \infty]$, we have the following decay estimate

$$\|D^\beta(u_c(t) - u_p(t))\|_{L^p} \leq C \min\left\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2-1/2}\right\} E_{|\beta|+[m/2]+1}$$

Euler equations with damping

$$\begin{cases} \rho_t + \mathrm{div}(\rho v) = 0, \\ (\rho v)_t + \mathrm{div}(\rho v \otimes v) + \frac{1}{\gamma}\nabla \rho^\gamma = -v \end{cases}$$

Linearize the system around the constant state $(\bar\rho, \bar v) = (1, 0)$.

$$\begin{cases} \rho_t + \mathrm{div}\, v = 0 \\ v_t + \nabla \rho = -v \end{cases}$$

$$\Downarrow$$

$$\|D^\beta(\rho(t) - 1)\|_{L^p} \leq C \min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2}\Big\}$$

$$\|D^\beta v\|_{L^p} \leq C \min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2-1/2}\Big\},$$

Euler equations with damping

$$\begin{cases} \rho_t + \mathrm{div}(\rho v) = 0, \\ (\rho v)_t + \mathrm{div}(\rho v \otimes v) + \frac{1}{\gamma}\nabla \rho^\gamma = -v \end{cases}$$

Linearize the system around the constant state $(\bar\rho, \bar v) = (1, 0)$**.**

$$\begin{cases} \rho_t + \mathrm{div}\, v = 0 \\ v_t + \nabla \rho = -v \end{cases}$$

$$\Downarrow$$

$$\|D^\beta(\rho(t) - \rho_w(t))\|_{L^p} + \|D^\beta(\rho(t) - \rho_p(t))\|_{L^p} \le C \min\left\{1, t^{-\frac{m}{2}(1-\frac{1}{p}) - |\beta|/2 - 1/2}\right\},$$

where ρ_w **and** ρ_p **are the solutions of**

$$\underbrace{\rho_{w,tt} - \Delta\rho_w + \rho_{w,t} = 0}_{\text{dissipative wave equation eq.}} \quad , \quad \underbrace{\rho_{p,t} - \Delta\rho_p = 0}_{\text{heat eq.}}$$

The Chapman–Enskog expansion (1D case)

We need to consider also the terms of the order of u^2
We can write the conservative equations as

$$\overbrace{u_{c,t} + \Big(A_{11}(0)u_c + \tilde{A}(u_c,u_c)\Big)_x + (A_{12}(0)(D_{u_d}q(0))^{-1}A_{21}(0))u_{c,xx}}^{\text{associated nonlinear parabolic part}} = \overbrace{S_x}^{\text{faster decay}}$$

where

$$\tilde{A} = \frac{1}{2}\left(L_0 D^2_{u_c} f(0) - A_{12}(0)(D_{u_d}q(0))^{-1} D^2_{u_c} q(0)\right)$$

Theorem 6 **Let u_p be the solution of the associated parabolic problem, for $m = 1$ and $p \in [1,\infty]$. For every $\mu \in [0,1/2)$, if E_1 sufficiently small with respect to $(1/2-\mu)$, then we have the following decay estimate**

$$\|D^\beta(u_c(t) - u_p(t))\|_{L^p} \le C \min\left\{1, t^{-\frac{1}{2}(1-\frac{1}{p})-\mu-\beta/2}\right\} F_{\beta+4}$$

p-system with relaxation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \\ \partial_t v + \partial_x \sigma(u) = h(u) - v \qquad \sigma'(u) > |h'(u)| \end{cases}$$

with $\lambda = \sqrt{\sigma'(0)}$ **and** $a = h'(0)$**, and** $\nu := \lambda^2 - a^2 > 0$

Chapman-Enskog approximation

$$u_{p,t} + h'(0)u_{p,x} + \frac{1}{2}h''(0)(u_p^2)_x - \nu u_{p,xx} = 0$$

Part III: Asymptotic High Order (AHO) schemes

D. Aregba-Driollet, M. Briani, R. Natalini, Asymptotic high-order schemes for dissipative hyperbolic systems, in preparation

Suppose u to be the solution of our problem with initial condition $u(x,0) = u_0(x)$ and let w be its asymptotic state.

Assume that there exists a constant $\gamma > 0$ such that,

$$\|u - w\| = O(t^{-\gamma}) \quad \textbf{as} \quad t \to +\infty.$$

Then, we call Asymptotic High-Order (AHO) schemes, those schemes that:

- **verify consistency and convergence properties,**
- **are high-order approximations to the asymptotic states of the problem,**

as a consequence, for the truncation error $\mathcal{T}$ holds

$$\boxed{\mathcal{T} \sim O(t^{-\gamma}) + O(h^{p}),}$$

where h is the mesh parameter and $p > 1$.

example: $u_t + au_x = -u \implies aw_x = -w$

▶ FOREFATHER $\rightsquigarrow$ **An Asymptotic 2-Order scheme (AHO2)**

upwinding of the source term (Roe 1986)

$$\frac{dv_j}{dt} + \overbrace{a\frac{v_j - v_{j-1}}{h}}^{\textbf{convection}} + \overbrace{\frac{v_j + v_{j-1}}{2}}^{\textbf{source}} = 0$$

This scheme is a second-order approximation to $aw_x + w = 0$

- **General scheme, $\tilde{S}_0$ linear difference operator,**

$$\frac{U^{n+1} - U^n}{\Delta t} + \tilde{S}_0 U^n = 0,$$

- **Truncation Error, u exact solution**

$$\mathcal{T}^n := \frac{u^{n+1} - u^n}{\Delta t} + \tilde{S}_0 u^n$$

- **AHO Truncation Error, w steady state**

$$\begin{array}{ccccccc}
\mathcal{T}_j^n & = & \dfrac{u_j^{n+1} - u_j^n}{\Delta t} & + & \tilde{S}_0(u_j^n - w_j) & + & \tilde{S}_0 w_j \\
 & & \wr & & \wr & & \wr \\
 & & u_t & & \frac{u-w}{h} & & h^p \\
 & & \downarrow & & \downarrow & & \downarrow \\
\textit{large-time} & = & 0 & + & 0 & + & \textbf{High Accuracy}
\end{array}$$

A linear p–system with dissipation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = au - v, \quad \lambda^2 > a^2 \end{cases}$$

The Conservative-Dissipative form

$$w_c = u, \; w_d = -b^{-1}(au - v), \qquad b = (\lambda^2 - a^2)^{1/2} > 0$$

$$\begin{cases} \partial_t w_c + a\partial_x w_c + b\partial_x w_d = 0 \\ \\ \partial_t w_d + b\partial_x w_c - a\partial_x w_d = -w_d \end{cases}$$

- **Shizuta-Kawashima** $\rightsquigarrow b \neq 0$
- **Subcharacteristic condition (always true)**

$$-\lambda = -\frac{1}{2}\sqrt{(2a)^2 + b^2} < a < \lambda = \frac{1}{2}\sqrt{(2a)^2 + b^2}$$

A general consistent scheme

$$\partial_t W + A\partial_x W = BW, \qquad W = (w_c, w_d)$$

$$W_j^{n+1} = W_j^n - \frac{k}{h} \quad A(W_{j+1}^n - W_{j-1}^n) + \frac{k}{2h} \overbrace{Q(W_{j+1}^n - 2W_j^n + W_{j-1}^n)}^{\text{artificial diffusion}}$$

$$+k(B_{-1}W_{j-1}^n + B_0 W_j^n + B_1 W_{j+1}^n)$$

$$Q = \overbrace{\begin{pmatrix} \lambda & 0 \\ & \\ 0 & \lambda \end{pmatrix}}^{\text{upwind}}, \qquad B_{-1} + B_0 + B_1 = B + hC$$

Stationary solution

$$AW_x = BW \Rightarrow \; W_x = A^{-1}BW \Rightarrow \; \partial_k W = (A^{-1}B)^k W$$

Truncation error

$$\begin{aligned} AW_x - BW \quad &= h(\tfrac{Q}{2} W_{xx} + (B_1 - B_{-1})W_x + CW) \\ & h^2(-\tfrac{A}{2} W_{xxx} + \tfrac{1}{2}(B_1 + B_{-1})W_{xx}) \\ & h^3(-\tfrac{Q}{22} W_{xxxx} + \tfrac{1}{6}(B_1 - B_{-1})W_{xxx}) + O(h^4) \end{aligned}$$

Three different schemes

- **pointwise (Euler): order 1**

$$B_1 = B_{-1} = 0,\ B_0 = B,\ C = 0$$

- **source upwinding (Roe): order 2**

$$B_{-1} = \frac{1}{4\lambda}\begin{pmatrix} 0 & -1 \\ 0 & a-\lambda \end{pmatrix}, \quad B_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \frac{1}{4\lambda}\begin{pmatrix} 0 & 1 \\ 0 & -(a+\lambda) \end{pmatrix}$$

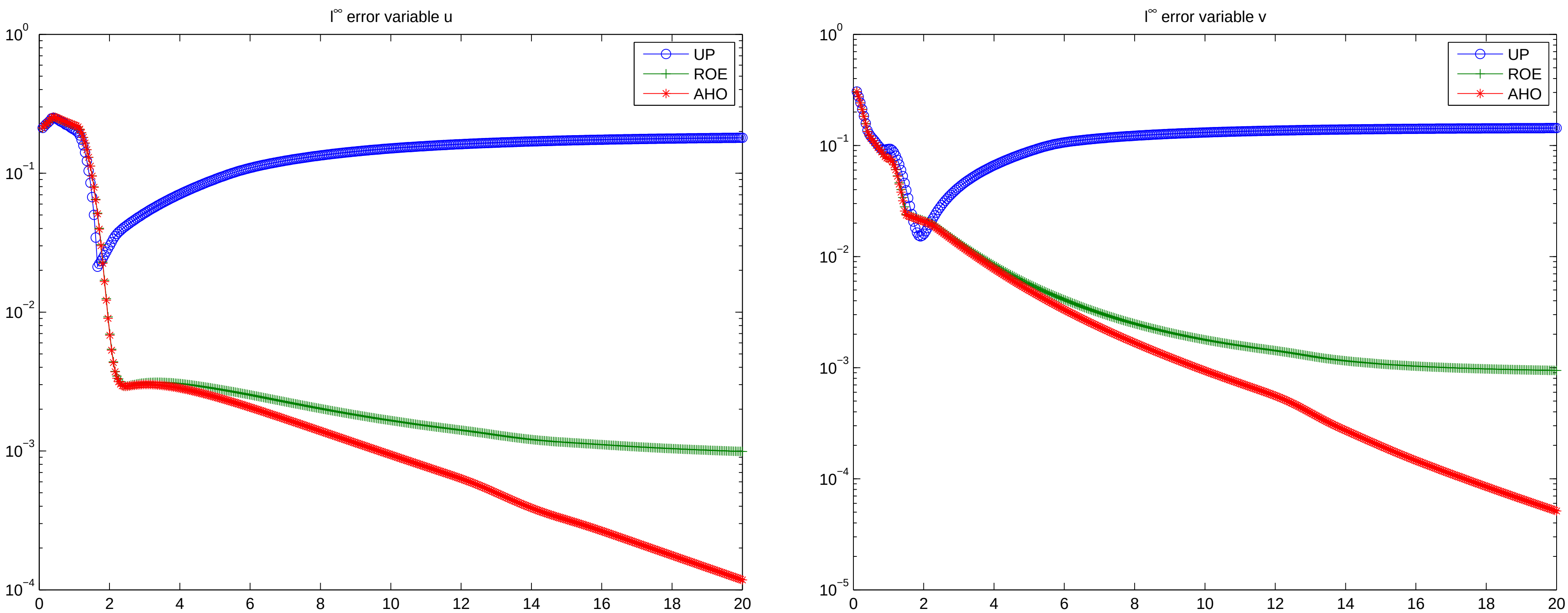
- **AHO4 (Aregba-Driollet, Briani, N.): order 4**

$$B_{-1} = \frac{1}{6}B - \frac{1}{8\lambda}\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}, \ B_0 = \frac{2}{3}B + \frac{ha}{4\lambda^3}\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}, \ B_1 = \frac{1}{6}B + \frac{1}{8\lambda}\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}$$

with

$$B_{-1} + B_0 + B_1 = B + hC, \text{ for } C = \frac{a}{4\lambda^3}\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}$$

L^∞ difference with the steady state solution



Comparison of upwind, Roe, and AHO4 schemes

Peturbation of a constant state

Chapman-Enskog limit

$$u_t + au_x = b^2 u_{xx}$$

Diffusive expansion of the schemes

- **pointwise (Euler)**

$$u_t + au_x = \left(b^2 + \frac{h}{2}\lambda\right) u_{xx}$$

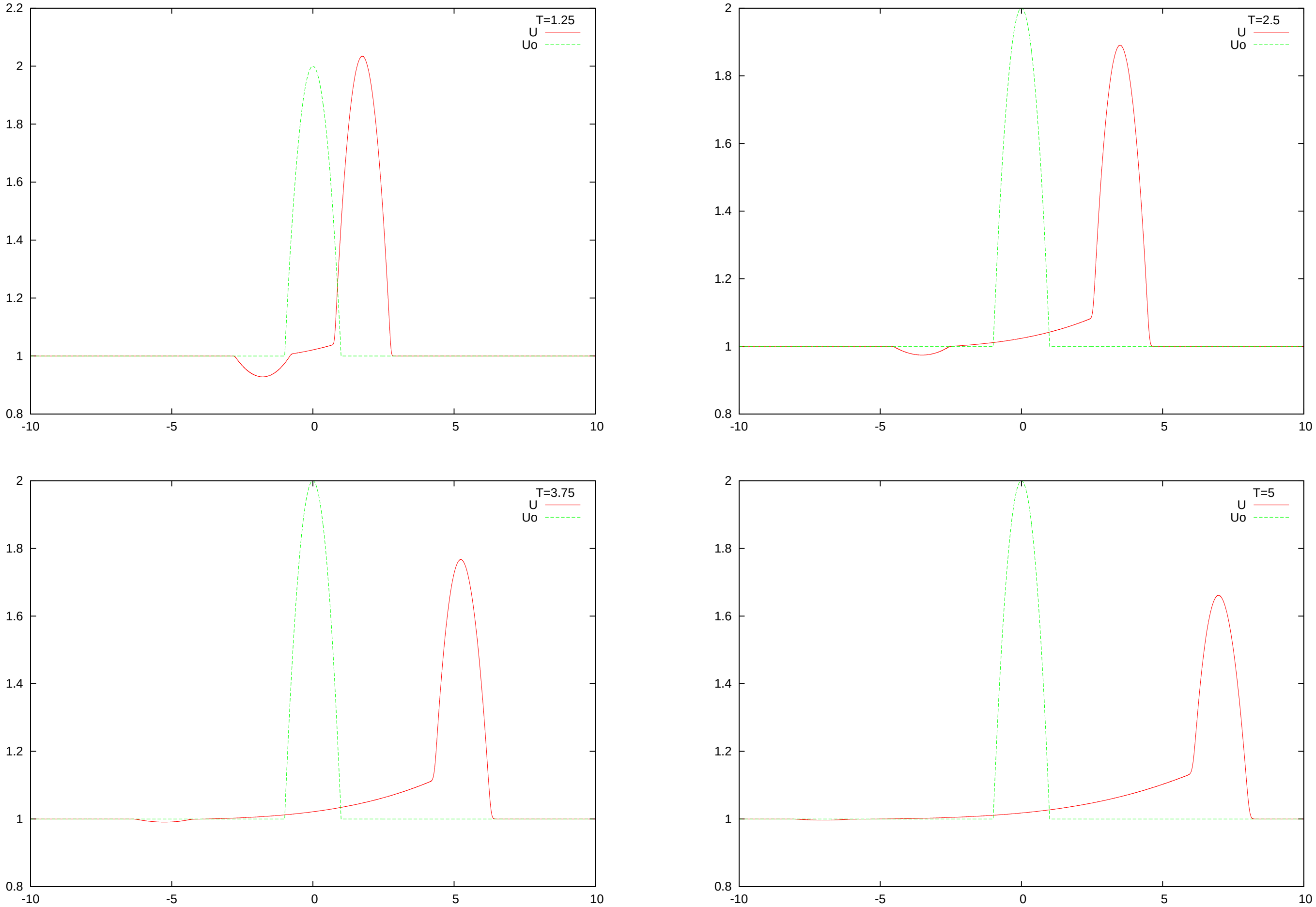
- **source upwinding (Roe)**

$$u_t + au_x = \left(b^2 + \frac{h}{2}(\lambda - \frac{1}{\lambda})\right) u_{xx}$$

- **AHO4 (Aregba-Driollet, Briani, N.)**

$$u_t + a\left(1 + h\frac{a}{4\lambda^3 + ha^2}\right) u_x = \left(b^2 - 2h\left(\frac{2\lambda^4 + a^2 + \lambda^2}{4\lambda^3 + ha^2}\right)\right) u_{xx}$$

Evolution of the solution vs. initial condition



Conclusions and open directions

- **main analytical result:** complete description of the asymptotic behavior of smooth solutions for dissipative hyperbolic systems in the multiD case (in L^p)
- **numerical schemes:** accurate approximations of long time regime
- **Open!** Global existence and behavior **without** the Shizuta-Kawashima condition (work in progress C. Mascia-R.N. for the 2×2 case in 1D)
- **Open!** Interaction of dispersive and dissipative effects in MultiD (where Shizuta-Kawashima condition is not in general verified)
- **Open!** Behavior (even just existence) of discontinuous solutions in the 1D case
- **Open!** AHO schemes for the general (1D) case