Finite Element Discretization of a Pseudo-parabolic Equation

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1 Introduction

We consider the following linear pseudo-parabolic problem :

$$u_t - \Delta u - \tau \Delta u_t = f \tag{1}$$

for $x \in \Omega, t \in J = (0,T], \tau > 0$, where Ω is a bounded domain in $\mathbb{R}^d$, with Lipschitz boundary $\partial\Omega$.
We consider homogenuous Dirichlet boundary condition for u, and the initial condition :

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2}$$

where $u_0 \in H_0^1(\Omega)$.

This model describes :

- the flow of fluids in a fissured porous medium (Barenblatt, Entov and Ryzhik 1990 [3])
- the two-phase flow in porous media with dynamical capillary pressure (Cuesta, Van Duijn, and Hulshof 1999 [4])

2 DGFEM Discretization

2.1 Motivation

Develop a numerical scheme with the following properties :

- **Robustness**
- **Maintain accuracy**
- **Provide a local, element based discretization which is suitable for** hp **mesh adaptation**
- **Easy to parallelize**

2.2 DG Time Discretization

Let $\mathcal{M}$ be a partition of $J =]0,T[$ into $N(\mathcal{M})$ subintervals $\{I_n\}_{n=1}^N$ given by $I_n =]t_{n-1}, t_n[$. The time step k_n is $k_n := t_n - t_{n-1}$. We defined the one-sided limits in $H := L^2(\Omega)$ (or $V := H_0^1(\Omega)$) of a function u as

$$u_n^+ = \lim_{s\to 0^+} u(t_n+s),\ 0 \le n \le N-1, \quad u_n^- = \lim_{s\to 0^+} u(t_n-s),\ 1 \le n \le N \tag{3}$$

and we set $[u]_n = u_n^+ - u_n^-, \quad 0 \le n \le N-1$.
The semidiscrete space in which we want to discretise (1)-(2) in time is

$$\mathcal{V}_k^r = \{u :\ J \to V :\ u|_{I_n} \in \mathcal{P}^r(I_n; V),\ 1 \le n \le N\}. \tag{4}$$

These functions are allowed to be discontinuous at the nodal points.

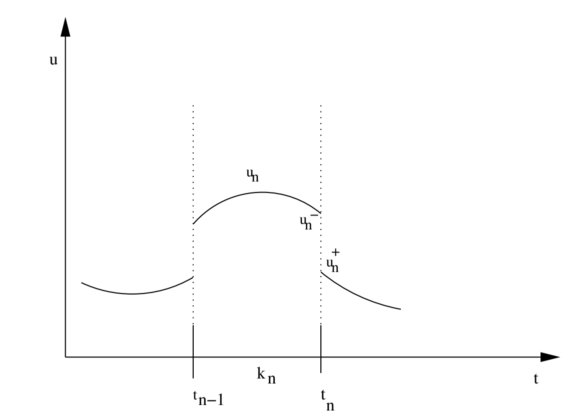

Fig 1: Discontinuous function u

We consider the following discontinuous Galerkin approximation of (1)-(2) : Find $u_k \in \mathcal{V}_k^r$ satisfies

$$a(u_k, v_k) = F(v_k)\ \ \forall\, v_k \in \mathcal{V}_k^r. \tag{5}$$

The forms a and F are given by

$$a(u,v) = \int_0^{t_N} \big((u_t, v) + (\nabla u, \nabla v) + \tau(\nabla u_t, \nabla v)\big)dt + \sum_{n=1}^{N-1}([u]_n, v_n^+)$$
$$+ (u_0^+, v_0^+) + \tau \sum_{n=1}^{N-1}([\nabla u]_n, \nabla v_n^+) + \tau(\nabla u_0^+, \nabla v_0^+), \tag{6}$$

and

$$F(v) = (u_0, v_0^+) + \tau(\nabla u_0, \nabla v_0^+) + \int_0^{t_N} (f, v)dt.$$

Remark : Owing to the discontinuity of the trial and test space the DGFEM, (5) can be interpreted as an implicit time marching scheme, where u_k is obtained by solving successively evolution problems on I_n for $n = 1, ..., N$ with initial values $u_{k,n-1}^-$, i.e if u_k is already given on $I_k,\ 1 \le k \le n-1$, we determine u_k on I_n by solving :
Find $u_k \in \mathcal{P}^r(I_n; V)$ such that

$$\int_{I_n} \{(u_k', v_k)_H + (\nabla(u_k + \tau u_k'), \nabla v_k)_H\}dt + (u_{k,n-1}^+, v_{k,n-1}^+)_H$$
$$+\tau(\nabla u_{k,n-1}^+, \nabla v_{k,n-1}^+)_H \tag{7}$$
$$= \int_{I_n} (f, v_k)_H dt + (u_{k,n-1}^-, v_{k,n-1}^+)_H + \tau(\nabla u_{k,n-1}^-, \nabla v_{k,n-1}^+)_H$$

for all $v_k \in \mathcal{P}^r(I_n; V)$. Here we set $u_{k,0}^- = u_0$.

2.2 Time shape functions and spatial problems

The DGFEM reduces the pseudo-parabolic equation (1) in each time step I_n to a coupled elliptic system of $r+1$ equations. We set $u_k = \sum_{j=0}^r u_{k,j}\varphi_j$ and $v_k = \sum_{i=0}^r v_{k,i}\psi_i$, $u_{k,j}, v_{k,i} \in V$, where $\{\varphi\}_{j=0}^r$ and $\{\psi\}_{j=0}^r$ are the normalised Legendre polynomials. Problem (7) is then equivalent to :
Find $\{u_{k,j}\}_{j=0}^r \subset V$ such that for all $\{v_{k,i}\}_{i=0}^r \subset V$

$$\sum_{i,j=0}^r \{[\int_I \varphi_j'\psi_i dt + \varphi_j^+(t_0)\psi_i^+(t_0)]\big((u_{k,j}, v_{k,i})$$
$$+\tau(\nabla u_{k,j}, \nabla v_{k,i})\big) + (\int_I \varphi_j\psi_i dt)(\nabla u_{k,j}, \nabla v_{k,i})\}$$
$$= \sum_{i=0}^r \{(\int_I f\psi_i dt, v_{k,i}) + \big((u_{init}, v_{k,i}) + \tau(\nabla u_{init}, \nabla v_{k,i})\big)\psi_i^+(t_0)\} \tag{8}$$

We introduce the matrices

$$\hat{A}_{ij} := \int_{-1}^1 \hat\varphi_j'\hat\psi_i d\hat t + \hat\varphi_j^+(-1)\hat\psi_i^+(-1), \quad \hat{B}_{ij} := \int_{-1}^1 \hat\varphi_j\hat\psi_i d\hat t,$$

Then (8) is equivalent to find $\{u_{k,j}\}_{j=0}^r \subset V$ such that for all $\{v_{k,i}\}_{i=0}^r \subset V$

$$\sum_{i,j=0}^r \hat{A}_{ij}\big((u_{k,j}, v_{k,i}) + \tau(\nabla u_{k,j}, \nabla v_{k,i})\big) + \frac{k}{2}\hat{B}_{ij}(\nabla u_{k,j}, \nabla v_{k,i})$$
$$= \sum_{i=0}^r \frac{k}{2}(\hat f_i^1, v_{k,i}) + (\hat f_i^2, v_{k,i}) \tag{9}$$

Remark 1 *The matrices $\hat A$ and $\hat B$ are independent of the time step and can be calculated in a preprocessing step. Their size depend of r.*

The ideal choice of time shape functions is $\{\varphi_i\}$ would be the one where $\hat A$ and $\hat B$ diagonalize simultaneously.

3 Existence and uniqueness of discret solution

3.1 Stability Lemma

Lemma 1 *For all $v_k,\ w_k \in \mathcal{V}_k^r$ there holds*

$$\begin{aligned} a(v_k, w_k) &= \sum_{n=1}^N \int_{I_n} \{-(v_k, w_k')_H + (\nabla v_k, \nabla(w_k - \tau w_k'))_H\}dt \\ &- \sum_{n=1}^{N-1}(v_{k,n}^-, [w_k]_n)_H + (v_{k,N}^-, w_{k,N}^-)_H \\ &- \tau\sum_{n=1}^{N-1}(\nabla v_{k,n}^-, [\nabla w_k]_n)_H + \tau(\nabla v_{k,N}^-, \nabla w_{k,N}^-)_H \end{aligned} \tag{10}$$

$$\begin{aligned} a(v_k, v_k) &= \sum_{n=1}^N \int_{I_n} \|\nabla v_k\|_H^2 dt + \frac{1}{2}\|v_{k,0}^+\|_H^2 + \frac{\tau}{2}\|\nabla v_{k,0}^+\|_H^2 \\ &+ \frac{1}{2}\sum_{n=1}^{N-1}\|[v_k]_n\|_H^2 + \frac{\tau}{2}\sum_{n=1}^{N-1}\|[\nabla v_k]_n\|_H^2 \end{aligned} \tag{11}$$

3.1 Existense and uniqueness

Proposition 1 *The DGFEM (5) has a unique solution $u_k \in \mathcal{V}_k^r$. Moreover if u is the solution to (1)-(2), one has the Galerkin orthogonality*

$$a(u - u_k, v_k) = 0 \ \ \forall\, v_k \in \mathcal{V}_k^r$$

4 Error Analysis

4.1 The interpolation Error

We introduce first the operator Π^r defined by :

Definition 1 *Let $I = (-1,1)$. For a function $u \in L^2(I;V)$ which is continuous at $t = 1$, we define $\Pi^r u \in \mathcal{P}^r(I;V)$, $r \in \mathbb{N}_0$, via the $r+1$ conditions*

$$\int_I (\Pi^r u - u, q)_\tau dt = 0\ \forall\, q \in \mathcal{P}^{r-1}(I;V), \quad \Pi^r u(1) = u(1) \text{ in } V. \tag{12}$$

where

$$(u,v)_\tau = (u,v)_H + \tau(\nabla u, \nabla v)_H,$$

Π^r is well defined. We set

$$\Pi^r u = \sum_{i=0}^{r-1} u_i L_i + (u(1) - \sum_{i=0}^{r-1} u_i)L_r. \tag{13}$$

we prove that π^r is unique and satisfy (12).

Definition 2 *On an arbitrary interval $I_n = (t_{n-1}, t_n)$, with $k_n := t_n - t_{n-1}$ we define $\Pi_{I_n}^r$ via the linear map $Q : (-1,1) \to I_n,\ \hat t \to t = \frac{1}{2}(t_{n-1} + t_n + \hat t k_n)$ as*

$$\Pi_{I_n}^r u = [\Pi^r(u \circ Q)] \circ Q^{-1}$$

Main Result
Now in order to give a priori estimate, we have to estimate the operator Π^r

Theorem 1 *Let $I_n = (t_{n-1}, t_n)$, $k_n := t_n - t_{n-1}$, $u \in H^{r_n+1}(I_n;V)$, $r_n \ge 1$. Then, there is a constant C independent of r and k such that*

$$\|u - \Pi_{I_n}^{r_n} u\|_{L^2(I_n;V)}^2 \le \Big(\frac{k_n}{2}\Big)^{2(r_n+1)} \frac{1}{r_n^4 r_n!^2}\|u^{(r_n+1)}\|_{L^2(I_n;V)}^2 \tag{14}$$

Proof : We proof the theorem in the reference element $I = (-1,1)$, then we use the transformation Q.

From the Definition of Π^r we have $\Pi^r u = \sum_{i=0}^{r-1} u_i L_i + (u(1) - P^r u(1))L_r$. Therefore,

$$u - \Pi^r u = (u - P^r u) - (u(1) - P^r u(1))L_r.$$

using the othogonality of Legendre polynomial we get

$$\|u - \Pi^r u\|_{L^2(I;V)}^2 = \|u - P^r u\|_{L^2(I;V)}^2 + \frac{2}{2r+1}(\|u(1) - (P^r u)(1)\|_V)^2$$

From [1], we have

$$\|u - P^r u\|_{L^2(I;V)}^2 \le \frac{1}{r^2(2r)!}\|u^{(r+1)}\|_{L^2(I;V)}^2, \tag{15}$$

For the second therm we use Darboux-Christoffel formula :

$$(P_r u)(s) = \frac{r+1}{2}\int_{-1}^1 \frac{L_{r+1}(s)L_r(t) - L_r(s)L_{r+1}(t)}{s-t}u(t)dt \tag{16}$$

and

$$\frac{r+1}{2}\int_{-1}^1 \frac{L_{r+1}(s)L_r(t) - L_r(s)L_{r+1}(t)}{s-t}dt = 1.$$

then we get in particular for $s = 1$

$$(P_r u)(1) - u(1) = \frac{r+1}{2}\int_{-1}^1 \frac{L_{r+1}(t) - L_r(t)}{t-1}(u(t) - u(1))dt$$

the result is obtain using the following Legendre polynomial propertie, then Cauchy-Schwarz

$$L_r = \frac{1}{2^r r!}\Big(\frac{d}{dt}\Big)^r(1-t^2)^r,$$

4.2 A priori error estimate

Proposition 2 *Let u be the solution of $(1.1) - (1.2)$ and u_k the solution of the DGFEM (5) in $\mathcal{V}_k^r$. Let $\mathcal{I}u \in \mathcal{V}_k^r$ be the interpolant of u which is defined on each time interval I_n as $\mathcal{I}u|_{I_n} = \Pi_{I_n}^r(u|_{I_n})$. Then there holds*

$$\|u - u_k\|_{L^2(I;V)} \le C\|u - \mathcal{I}u\|_{L^2(I;V)}.$$

The constant C is in particular independent of T.

Proof: the assertion follows by using the properties of the operator Π^r, the orthogonality propertie and Cauchy Schwartz.
Proposition 2 and Theorem 1 give error estimates for the DGFEM (5) which are valid if the exact solution is at least in $H^1(I;V)$

Theorem 2 *Let u be the solution of (1)-(2) and u_k the solution of the DGFEM (5). Assume that $u|_{I_n} \in H^{r_n+1}(I_n;V)$ for $0 \le n \le N$. Then*

$$\|u - u_k\|_{L^2(I_n;V)}^2 \le C\sum_{n=1}^N \Big(\frac{k_n}{2}\Big)^{2(r_n+1)}\frac{1}{r_n^4 r_n!^2}\|u^{(r_n+1)}\|_{L^2(I_n;V)}^2 \tag{17}$$

C is a constant independent of r and k

4.3 A posteriori error estimate :

5 Numerical Tests

Let $\Omega = (0,1)^2$ and $J = (0, 0.1)$. We choose $u_0(x,y) = \sin(\pi x)\sin(\pi y)$ and $f = -\exp(-t)\sin(\pi x)\sin(\pi y)$, with $\tau = 1$. u_0 is actually the first eigenfunction of the Laplacian and $u_0 \in H_0^1(\Omega)$. The corresponding exact solution $u(t,x,y)$ is smooth in space and time and given by :

$$u(t,x,y) = \exp(-t)\sin(\pi x)\sin(\pi y).$$

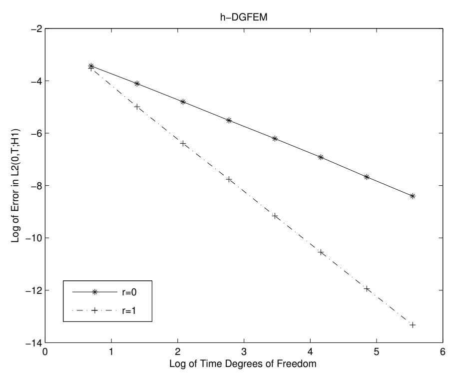

Fig 2: Convergence rate for h-DGFEM

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