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Modelling of free boundary problems in thin films mechanics: a multifluid approach

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Summary

Hydrodynamic lubrication is concerned with the behaviour of thin film flows between two close surfaces in relative motion. For a long time it has been observed, in most of lubrication regimes, a free boundary between a full film area (filled with oil, for instance) and another area which contains a mixture of liquid-gas at the vapor pressure. One of the most usable models to describe such phenomena is a pressure-saturation problem (Elrod-Adams model). This reference model has been obtained in a very heuristic way and, here, we present some attempt to derive it in a more rigorous way, using a bifluid approach. For this, we use the bifluid Stokes equations in thin films. The bifluid model includes shear effects leading to a non classical problem. Thus, we present new mathematical results on this model. We describe the large-time behaviour of its solution and we compare the (stationary) solution of the bifluid model to the solution of the Elrod-Adams model.

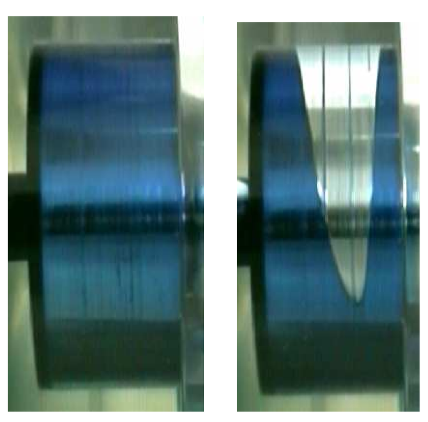
1. Introduction to cavitation phenomena

The Reynolds equation has been used for a long time to describe the behaviour of a viscous flow between two close surfaces in relative motion. The transition of the Stokes equation to the Reynolds equation has been proved by Bayada and Chambat in [1].

• OBSERVATION - Actually, the pressure p may decrease until reaching the vapor pressure p_s so that two different parts exist in the device:

▷ the satured regions: $p > p_s$ and the Reynolds equation is valid.

▷ the cavitated regions: $p = p_s$ and the Reynolds equation is NOT valid. There is a mixture of liquid-gas in the device (see the following pictures).



• MATHEMATICAL MODEL - Cavitation is classically described by the Elrod-Adams model, which is the reference model in tribology:

$$(\mathcal{P}_1)\begin{cases} \text{div}\left(\dfrac{h^3}{6\mu_l}\nabla p\right) = v_0 \dfrac{\partial}{\partial x_1}\Big(\theta\, h\Big), \\ p \geq 0, \qquad \theta \in H(p), \end{cases}$$

where h is the gap between the two surfaces, μ_l the (liquid) lubricant viscosity and v_0 the relative velocity of the surfaces; p is the pressure distribution, θ a saturation function which denotes the local ratio of the liquid phase. Thus, one has:

▷ in the satured regions, $p > 0$, $\theta = 1$ (the classical Reynolds equation is recovered).

▷ in cavitated regions, $p = 0$, $0 \leq \theta \leq 1$ (partial lubrication).

However, the introduction of such a saturation function in the lubrication problem is mainly heuristic. Thus, in order to justify this model, we focus on bifluid models in thin films: the idea is to consider the flow of a gas / liquid mixture in lubrication regimes.

• A BIFLUID MODEL IN THIN FILMS [8, 9] - Starting from a Stokes flow model of two viscous fluids [6] characterized by two different viscosities μ_l and $\mu_g < \mu_l$, and introducing the saturation s of the reference (liquid) fluid, one has the following result: under some assumption on the shape of the free boundary which separates the two phases, the asymptotic (thin film) equations are:

$$(\mathcal{P}_2)\begin{cases} \dfrac{\partial}{\partial x}\left(A(s)\dfrac{h^3(x)}{6\mu_l}\dfrac{\partial p}{\partial x}\right) = v_0 \dfrac{\partial}{\partial x}\left(B(s)h(x)\right) & \sim \textbf{Reynolds Eq.} \\ h(x)\dfrac{\partial s}{\partial t} + \dfrac{\partial}{\partial x}\Big(Q_{in}(t) f(s) + v_0\, h(x)\, g(s)\Big) = 0 & \sim \textbf{Buckley-Leverett Eq.} \end{cases}$$

with appropriate initial boundary conditions. Functions f, g, A et B depend on the viscosity ratio $M = \mu_g/\mu_l$. Q_{in} denotes the input flow.

Question: From this bifluid model $(\mathcal{P}_2)$ (with, for instance, $M \sim 10^{-3}$ corresponding to liquid-gas mixture), is it possible to recover the Elrod-Adams model $(\mathcal{P}_1)$?

Remarks: The bifluid model is a pressure-saturation formulation (as the Elrod-Adams model). But, unlike this reference model, it is not stationary. Moreover, its derivation is not fully rigorous. Thus, our purpose contains the following steps:

(i) we state that this (asymptotic) bifluid model is well-posed,

(ii) we study the large-time behaviour of the solution of the bifluid model,

(iii) we compare the (stationary) solution of the bifluid model to the solution of the Elrod-Adams model.

2. Mathematical analysis of the bifuid model [4] (S. Martin)

■ **Step 1 - Reduction of the initial problem:** The Buckley-Leverett equation may be reduced to a scalar conservation law which is conservative w.r.t. s instead of $h\,s$: using an appropriate change of variables $(x,t) \mapsto (Y(x), \mathcal{T}(t))$, the Buckley-Leverett equation becomes

$$\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial y}\Big(f(u) + H(\tau,y)\,g(u)\Big) = 0, \quad \textit{with} \quad H(\tau,y) = \frac{v_0\,\left(h\circ Y^{-1}\right)(y)}{\left(Q_{in}\circ \mathcal{T}^{-1}\right)(\tau)}$$

the unknown u playing the role of a saturation: $u(\mathcal{T}(t), Y(x)) = s(t,x)$. In this setting, initial and boundary conditions are adapted in the same way.

■ **Step 2 - Study of the reduced problem:**

• *Assumptions:*

(i) $u_0, \bar{u} \in L^\infty(\Omega;[0,1]) \times L^\infty(\mathbb{R}^+;[0,1])$, (iii) $f,\ g \in C^1([0,1])$,

(ii) H is "regular" and bounded, (iv) $f(0)=0,\ f(1)=1,\ g(0)=g(1)=0$.

• *S.C.L. with a* **non-autonomous flux**, *on a* **bounded** *domain, with* L^∞ **data**:

$$(\mathcal{P}_r)\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(f(u) + H(t,x)\,g(u)\right) = 0, & (t,x) \in (0,+\infty)\times(0,1) \\ u(0,\cdot) = u_0, & x \in (0,1) \\ \text{“}u = \bar{u}\text{”}, & (t,x) \in (0,+\infty)\times\partial]0,1[\end{cases}$$

(Definition) - A function $u \in L^\infty((0,+\infty)\times(0,1))$ is a *weak entropy solution of* $(\mathcal{P}_r)$ if

$$\int_0^1\int_0^{+\infty}\left\{(u-\kappa)^{\pm}\frac{\partial\varphi}{\partial t} + \left(\Phi^{\pm}_{[f]}(u,\kappa) + H\Phi^{\pm}_{[g]}(u,\kappa)\right)\frac{\partial\varphi}{\partial x} - \text{sgn}_{\pm}(u-\kappa)\frac{\partial H}{\partial x}g(\kappa)\varphi\right\}dx\,dt$$
$$+\int_0^1 (u_0(x)-\kappa)^{\pm}\varphi(0,x)\,dx + \mathcal{L}\int_{\partial]0,1[}\int_0^{+\infty}(\bar{u}(t,r)-\kappa)^{\pm}\varphi(t,r)\,d\gamma(r)\,dt \geq 0,$$

for all $\kappa \in [0,1]$, $\varphi \in \mathcal{D}([0,+\infty[\times[0,1])$, $\varphi \geq 0$. Here, $\mathcal{L}$ is the uniform Lipschitz constant of the flux and we use the concept of **"semi-Kružkov entropies-flux"** [7, 10] defined by

$$\begin{aligned} (u-\kappa)^{+} &= \max(u-\kappa,0), & \Phi^{\pm}_{[f]}(u,\kappa) &= \text{sgn}_{\pm}(u-\kappa)(f(u)-f(\kappa)) \\ (u-\kappa)^{-} &= (\kappa-u)^{+}, & \Phi^{\pm}_{[g]}(u,\kappa) &= \text{sgn}_{\pm}(u-\kappa)(g(u)-g(\kappa)). \end{aligned}$$

(Theorem) - $(\mathcal{P}_r)$ admits a unique weak entropy solution $u \in L^\infty(Q;[0,1])$.

(Corollary) - The bifluid problem is well-posed.

Sketch of the proof:

① Existence: vanishing viscosity method and regularization of the data,

② Uniqueness: doubling variables method.

Remark on the stability result:

Let $a = \min(\inf u_0, \inf \bar{u})$, $b = \max(\sup u_0, \sup \bar{u})$ with $[a,b] \subset [0,1]$. Then,

▷ in the autonomous case, $a \leq u \leq b$,

▷ in the non-autonomous case, $0 \leq u \leq 1$ (due to the properties of f and g).

Remark: Generalization to first order quasilinear equations on a bounded domain, with L^∞ data, for any space dimension $d \geq 1$.

3. Large-time behaviour of the solution [5] (S. Martin, J. Vovelle)

Although many works are devoted to the large-time behaviour of the entropy solution of SCL on unbounded domains, very few papers exist in the framework of bounded domains.

■ **Study of a simplified model:**

$$\begin{cases} \partial_t u + \partial_x\left(A(x,u)\right) = 0, & \text{on } (0,+\infty)\times(0,1), \\ u(\cdot,0) = u_0, & \text{on } (0,1), \\ u = \bar{u}, & \text{on } (0,+\infty)\times\partial(0,1) \end{cases}$$

The flux $A \in \mathcal{C}^1([0,1]^2)$ satisfies the following properties:

(i) $\forall x \in [0,1],\ A(x,\cdot) = A(1-x,\cdot),\ A(x,0) = 0,\ A(x,1) = Q \geq 0,$

(ii) $\forall u \in [0,1],\ \partial_x A(\cdot,u) \leq 0$ on $[0,1/2]$.

(iii) $\forall x \in [0,1],\ \exists\,\alpha(x) \in (0,1],\ \begin{cases}\partial_u A(x,\cdot) > 0 \text{ on } [0,\alpha(x)), \\ \partial_u A(x,\cdot) < 0 \text{ on } (\alpha(x),1],\end{cases}$

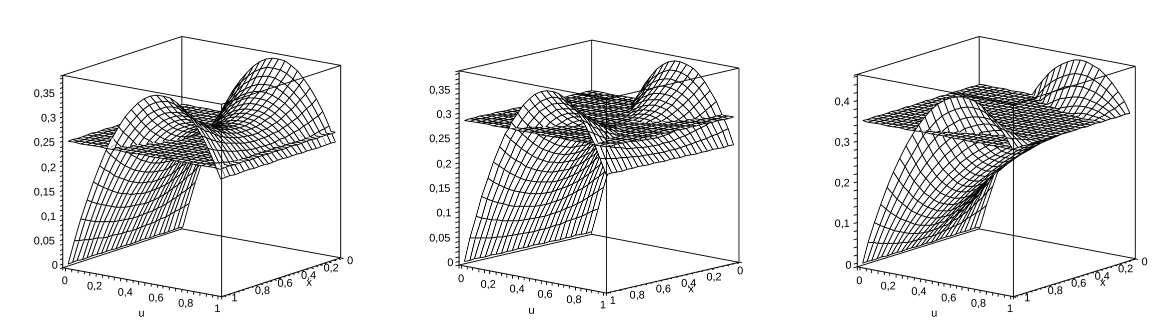
Let us define: $q_{\lim} := \min\{A(x,\alpha(x)),\ x \in [0,1]\} = A(1/2,\alpha(1/2))$.

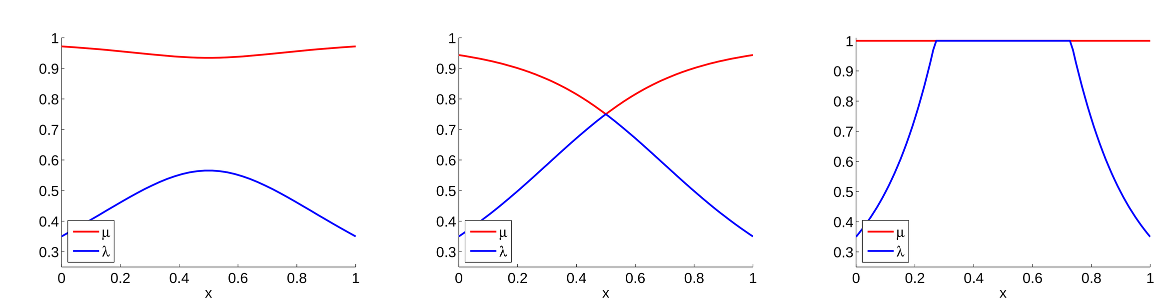
■ **Structure of the stationary solutions:** let us focus on solutions of

$$A(x,w(x)) = q \in \mathbb{R}, \quad x \in [0,1], \quad w : [0,1] \to [0,1] \text{ continuous}$$

According to the data $(Q,\ q_{\lim})$ and the value of q, this equation admits

0 solution / 1 solution / 2 solutions λ et μ with $\lambda \leq \mu$.



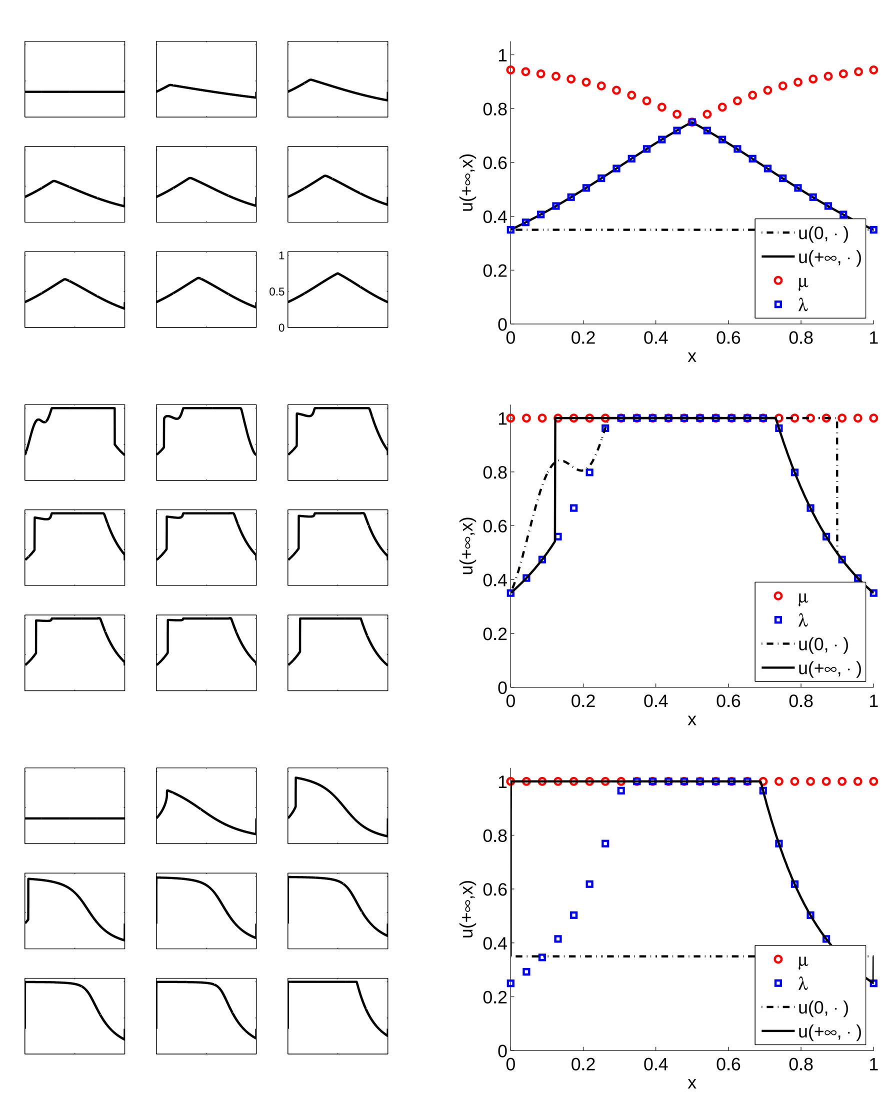


(Theorem)

- For all $u_0 \in L^\infty(0,1;[0,1])$, $\bar{u} \in [0,1]$, the entropy solution converges to a stationary state (described previously), with the possibility of a discontinuity (stationary entropy shock), the constant q being fixed by $q = \min(A(0,\bar{u}), q_{\lim})$.
- Moreover, in the case $\lambda \neq \mu$, then we obtain:
  - ▷ if $u_0 \leq \lambda$ (resp. $u_0 \geq \mu$), then u converges to λ (resp. μ),
  - ▷ if $\lambda \leq u_0 \leq \mu$ and $\bar{u} = \lambda(0)$ (resp. $\bar{u} = \mu(0)$), then there exists a stationary shock whose position is explicitely characterized.

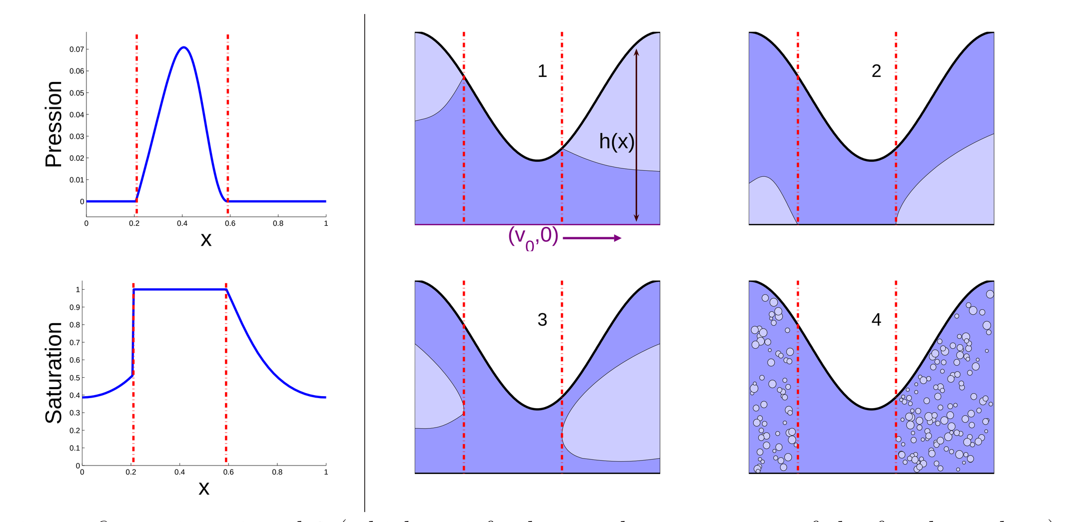
Sketch of the proof, based on [3]. Let $X_0 := L^\infty(0,1;[0,1])$ equipped with the L^1-norm topology. $S_{\bar{u}}(t) := S(\cdot,\bar{u})$ denotes the semi-group which, for all $u_0 \in X_0$, associates the value $u(t) \in X_0$ at time t of the entropy solution corresponding to the data $(u_0,\bar{u})$. Then we study the behaviour of the trajectories $S_{\bar{u}}(t)u_0$ for $t \to +\infty$.

■ **Some numerical illustrations:** $A(x,u) = Q\,u + H(x)\,u\,(1-u)$.



4. Numerical validation [2] (G. Bayada, S. Martin, C. Vázquez)

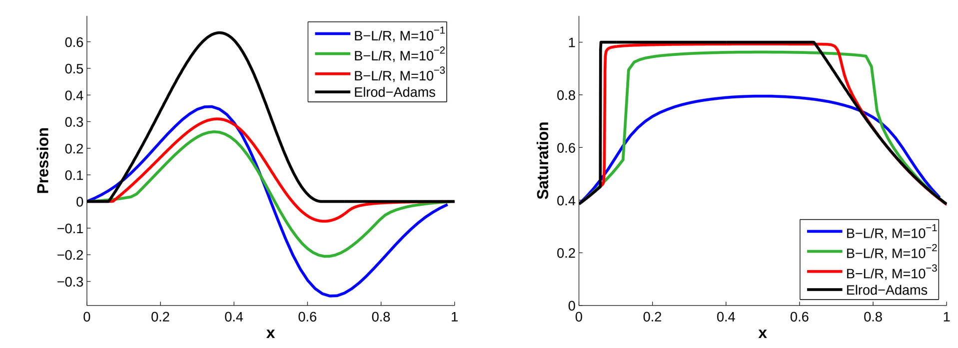
This section aims at comparing the solution of the Elrod-Adams model and the (stationary) solution of the bifluid model, with appropriate numerical and physical data. The bifluid model has been derived under the assumption that the free boundary is a graph... Actually, we do not know the structure of the mixture: bi-layer (1,2), multi-layer (3), bubbles (4)... ?



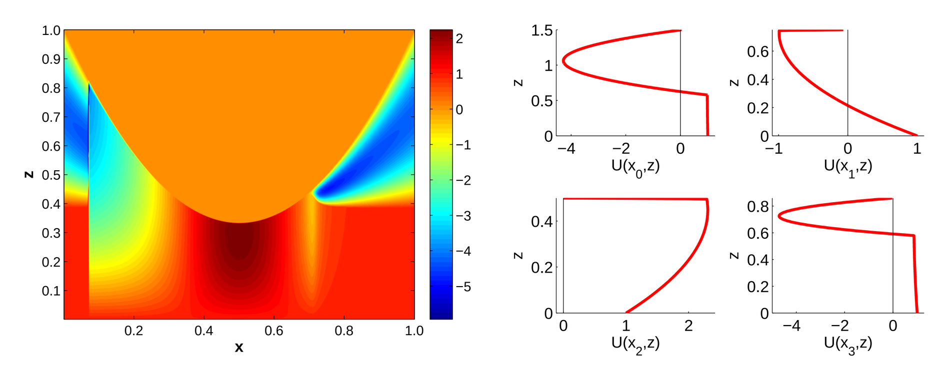
But, in configurations 1 and 2 (which satisfy the graph assumption of the free boundary), we can compare the solution of the Elrod-Adams model to the solution of the bifluid model for different values of M, with a particular interest for $M = 10^{-3}$ (liquid-gas mixture).

■ **Configuration 1:** the liquid (reference fluid) is adhering to the moving ("lower") surface.

▷ *Pressure and saturation profiles:*

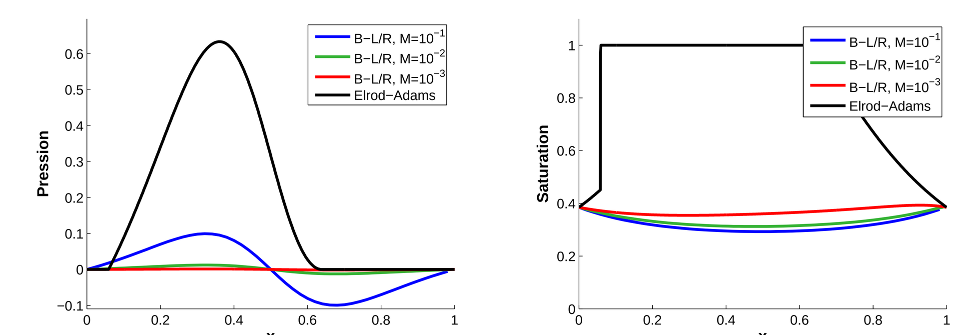


▷ *Horizontal velocity profiles,* for the bifluid model ($M = 10^{-3}$)



With $M = 10^{-3}$, saturation profiles of the two models are very close. Still, the bifluid saturation never reaches the value 1: a gas layer damps all the shear effects (see the velocity profiles) and leads to some error on the peak pressure. However, negative pressures tend to disappear; the film rupture may be predicted and analysed as an entropy stationary shock.

■ **Configuration 2:** the liquid (reference fluid) is adhering to the fixed ("upper") surface.



This geometrical assumption cannot be valid: the gas layer, between the liquid phase and the moving surface, damps all the shear efects: the liquid (reference) phase does not "see" the shear velocity of the device.

5. Conclusions

Difficulties for a complete validation of the bifluid approach seem to be related to the (adhering) boundary conditions which have been chosen in the initial bifluid Stokes flow: starting from a bilayer configuration, the saturation cannot reach the value 1, which leads to the differences in the pressure profiles. Thus, other boundary conditions should be considered...

Other bifluid models could be taken into account (diffusive interface, compressible fluids...).

It seems that the two models tend to coincide as M tends to 0... But the flux functions of the Buckley-Leverett equation degenerate (numerically, the CFL condition tends to explode). Formally, the limit problem is based on a scalar conservation law with a multivalued (non-autonomous) flux function... This field has never been studied.

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