Existence, unicity and asymptotic analysis for solution of the Burgers equation with relaxation

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1 Problem setting

The wave propagation processes in non-linear media with relaxation are described by the following integro-differential equation[1]:

$$\frac{\partial u}{\partial x} - A\frac{\partial^2 u}{\partial y^2} = \alpha\{c\frac{\partial f(u)}{\partial y} - \frac{\varepsilon}{\tau}u + \frac{\varepsilon}{\tau^2}\int_{-\infty}^{y} u(x,y')\exp\frac{y'-y}{\tau}dy'\}, \quad (1)$$

set in the domain $Q_T = \{(x,y) : 0 \le x \le T, -\infty < y < \infty\}$ with the initial condition

$$u(x=0,y) = \varphi(y) \quad (2)$$

and the periodicity condition

$$u \text{ — 1-periodic in } y. \quad (3)$$

[1] O .V. Rudenko, S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics*, Moscow: Nauka, 1975

Here A, c, ε, τ are constants,
$A > 0$, $\varepsilon > 0$, $\tau > 0$,

f, φ are functions,
$f \in C^1(\mathbb{R})$,
$\varphi \in W_2^{1,per}$,
where $W_2^{1,per}$ is a completion of the set $C^\infty_{1-per.}(\mathbb{R})$ of 1-periodic infinitely differentiable functions with respect to the norm $\| \cdot \|_2^1$,

$$\|\varphi(y)\|_2^1 = \left(\int_0^1 \varphi^2 + (\frac{\partial \varphi}{\partial y})^2 dy \right)^{\frac{1}{2}}.$$

By definition, set

$$\|u(x,y)\|_X^2 = \sup_{x \in [0,T]} \int_0^1 \left\{ u^2(x,y) + \left(\frac{\partial u}{\partial y} \right)^2 \right\} dy.$$

Def. *Let X be the completion of $C^\infty_{1-per.(y)}(\mathbb{R}^2)$ with respect to this norm.*

2 uxiliary problem

Consider the following equation

$$\frac{\partial u}{\partial x} - A\frac{\partial^2 u}{\partial y^2} = F(x,y), \tag{4}$$

where $F \in L^2(Q_t)$, whith conditions (2) and (3).

THEOREM 1. *Problem* (4), (2), (3) *has a unique solution in* X *for* $F \in L^2(Q_T)$, $\varphi \in W_2^{1,per}$ *such that* $< u >= 0$.

This solution satisfies the inequality

$$\|u\|_X \leq C(\|\varphi\|_2^{\frac{1}{2}} + \|F\|_{L^2}). \tag{5}$$

3 Estimate for the variation of the right hand side

Assume

$$F(u) = c\frac{\partial f(u)}{\partial y} - \frac{\varepsilon}{\tau}u + \frac{\varepsilon}{\tau^2}\int_{-\infty}^{y} u(x, y') \exp\frac{y' - y}{\tau} dy'.$$

THEOREM 2. *If $f'(u)$ satisfies a Lipschitz condition with constant L:*
$|f'(u_1) - f'(u_2)| \leq L|u_1 - u_2|$ *and if*
$\exists N = \sup_{u \in X} f'(u)$

then the following inequality holds:

$$\|F(u_1) - F(u_2)\|_{L^2} \leq K\|u_1 - u_2\|_X, \qquad (6)$$

where K is a constant.

4 Contraction operator Φ

We consider the problem:

$$Lu_1 = \alpha(F(u_0) + g),$$
$$u_1(x = 0, y) = 0,$$
$$u_1 \text{ — 1-per. by } y,$$

where $Lu = \dfrac{\partial u}{\partial x} - A\dfrac{\partial^2 u}{\partial y^2}$, $\alpha > 0$.

Determine operator

$$\Phi : X \to X, u_0 \mapsto u_1.$$

THEOREM 3. *There exists* $\alpha > 0$ *such that* $\Phi : B_R(X) \to B_R(X)$ *- is a contraction operator in the norm* $\|\cdot\|_X$.

5 Existence and unicity of solution of problem (1), (2), (3)

Using the fixed point theorem we obtain the following result.

THEOREM 4. *If $f'(u)$ satisfies a Lipschitz condition with constant L:*
$|f'(u_1) - f'(u_2)| \leq L|u_1 - u_2|$ *and if*
$\exists N = \sup_{u \in X} f'(u)$,

there exists $\alpha > 0$ such that problem (1), (2), (3) *has a unique solution in X.*

6 priori estimate

Consider two following problems:

$$\begin{aligned} &\frac{\partial u_i}{\partial x} - A\frac{\partial^2 u_i}{\partial y^2} = \alpha(F + g_i), \\ &u_i\big|_{x=0} = 0, \\ &i = 1, 2. \end{aligned} \tag{7}$$

THEOREM 5. *Choose* α *such that*

$$\alpha C \max(K, 1) \leq 1,$$

then the estimate holds:

$$\|u_\infty^1 - u_\infty^2\|_X \leq \frac{1}{1 - \alpha C \max(K, 1)} \|g_1 - g_2\|_{L^2}. \tag{8}$$

7 symptotic expansion

Suppose $\tau = \tau_0$, $\varepsilon \to 0$. We construct an asymptotic expansion of u as $\varepsilon \to 0$.

Let us introduce the following notation:

$$u^{(k)} = \sum_{i=0}^{k} \varepsilon^i u_i(x, y).$$

Functions $u_i(x, y)$ are satisfying the following *linear* differential equations.

Assume

$$\begin{cases} \dfrac{\partial u_0}{\partial x} - A\dfrac{\partial^2 u_0}{\partial y^2} = \alpha\{c\dfrac{\partial}{\partial y} f(u_0)\}, \\ \dfrac{\partial u_i}{\partial x} - A\dfrac{\partial^2 u_i}{\partial y^2} - \alpha\{c\dfrac{\partial}{\partial y}(f'(u_0)u_i)\} = \\ \alpha(c\displaystyle\sum_{p=2}^{i} \dfrac{\partial}{\partial y}(\dfrac{f^{(p)}(u_0)}{p!} \sum_{1\leq s_1 \ldots s_p \leq i}^{s_1+\ldots+s_p=i} u_{s_1}\ldots u_{s_p}) \\ -\dfrac{1}{\tau}u_{i-1} + \dfrac{1}{\tau^2}\displaystyle\int_{-\infty}^{y} u_{i-1}(x,y')e^{\frac{y'-y}{\tau}}dy'), \\ u_i\big|_{x=0} = \delta_{i0}\phi(y), \quad i = 0,1,2,\ldots \end{cases} \tag{9}$$

THEOREM 6. *The inequality holds:*

$$\|u^{(k)} - u\|_X \leq C\varepsilon^{k+1},$$

i.e. the asymptotic developpement $u^{(k)}$ converges to the exact solution u with respect to the norm $\|\cdot\|_X$ as $\varepsilon \to 0$ with the order ε^{k+1}.

8 Projects

We can consider problem (1), (2), (3) with the non-constant coefficients. If the parameters of the problem depend on some small variable δ, i.e. are fast oscillating, the equations describe some stratified heterogeneous media. In this case it's interesting to study the asymptotic behavior of the solutions as $\delta \to 0$ and try to obtain the corresponding homogenized problem.

9 References

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