

**Existence, unicity and
asymptotic analysis for
solution of the Burgers
equation with relaxation**

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1 Problem setting

The wave propagation processes in non-linear media with relaxation are described by the following integro-differential equation¹:

$$\frac{\partial u}{\partial x} - A \frac{\partial^2 u}{\partial y^2} = \alpha \left\{ c \frac{\partial f(u)}{\partial y} - \frac{\varepsilon}{\tau} u + \frac{\varepsilon}{\tau^2} \int_{-\infty}^y u(x, y') \exp \frac{y' - y}{\tau} dy' \right\}, \quad (1)$$

set in the domain

$Q_T = \{(x, y) : 0 \leq x \leq T, -\infty < y < \infty\}$ with the initial condition

$$u(x = 0, y) = \varphi(y) \quad (2)$$

and the periodicity condition

$$u \text{ — 1-periodic in } y. \quad (3)$$

¹O .V. Rudenko, S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics*, Moscow: Nauka, 1975

Here A, c, ε, τ are constants,

$A > 0, \varepsilon > 0, \tau > 0,$

f, φ are functions,

$f \in C^1(\mathbb{R}),$

$\varphi \in W_2^{1,per},$

where $W_2^{1,per}$ is a completion of the set

$C_{1-per.}^\infty(\mathbb{R})$ of 1-periodic infinitely differentiable

functions with respect to the norm $\|\cdot\|_2^1,$

$$\|\varphi(y)\|_2^1 = \left(\int_0^1 \varphi^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 dy \right)^{\frac{1}{2}}.$$

By definition, set

$$\|u(x, y)\|_X^2 = \sup_{x \in [0, T]} \int_0^1 \left\{ u^2(x, y) + \left(\frac{\partial u}{\partial y} \right)^2 \right\} dy.$$

Def. Let X be the completion of $C_{1-per.(y)}^\infty(\mathbb{R}^2)$ with respect to this norm.

2 auxiliary problem

Consider the following equation

$$\frac{\partial u}{\partial x} - A \frac{\partial^2 u}{\partial y^2} = F(x, y), \quad (4)$$

where $F \in L^2(Q_t)$, with conditions (2) and (3).

THEOREM 1. *Problem (4), (2), (3) has a unique solution in X for $F \in L^2(Q_T)$, $\varphi \in W_2^{1,per}$ such that $\langle u \rangle = 0$.*

This solution satisfies the inequality

$$\|u\|_X \leq C(\|\varphi\|_2^1 + \|F\|_{L^2}). \quad (5)$$

3 Estimate for the variation of the right hand side

Assume

$$F(u) = c \frac{\partial f(u)}{\partial y} - \frac{\varepsilon}{\tau} u + \frac{\varepsilon}{\tau^2} \int_{-\infty}^y u(x, y') \exp \frac{y' - y}{\tau} dy'.$$

THEOREM 2. *If $f'(u)$ satisfies a Lipschitz condition with constant L :*

$$|f'(u_1) - f'(u_2)| \leq L|u_1 - u_2| \text{ and if}$$

$$\exists N = \sup_{u \in X} f'(u)$$

then the following inequality holds:

$$\|F(u_1) - F(u_2)\|_{L^2} \leq K \|u_1 - u_2\|_X, \quad (6)$$

where K is a constant.

4 Contraction operator Φ

We consider the problem:

$$Lu_1 = \alpha(F(u_0) + g),$$

$$u_1(x = 0, y) = 0,$$

$$u_1 \text{ --- 1-per. by } y,$$

where $Lu = \frac{\partial u}{\partial x} - A \frac{\partial^2 u}{\partial y^2}$, $\alpha > 0$.

Determine operator

$$\Phi : X \rightarrow X, u_0 \mapsto u_1.$$

THEOREM 3. *There exists $\alpha > 0$ such that $\Phi : B_R(X) \rightarrow B_R(X)$ - is a contraction operator in the norm $\| \cdot \|_X$.*

5 Existence and unicity of solution of problem (1), (2), (3)

Using the fixed point theorem we obtain the following result.

THEOREM 4. *If $f'(u)$ satisfies a Lipschitz condition with constant L :*

$$|f'(u_1) - f'(u_2)| \leq L|u_1 - u_2| \text{ and if}$$

$$\exists N = \sup_{u \in X} f'(u),$$

there exists $\alpha > 0$ such that

problem (1), (2), (3) has a unique solution in X .

6 priori estimate

Consider two following problems:

$$\begin{aligned} \frac{\partial u_i}{\partial x} - A \frac{\partial^2 u_i}{\partial y^2} &= \alpha(F + g_i), \\ u_i|_{x=0} &= 0, \\ i &= 1, 2. \end{aligned} \tag{7}$$

THEOREM 5. *Choose α such that*

$$\alpha C \max(K, 1) \leq 1,$$

then the estimate holds:

$$\begin{aligned} \|u_\infty^1 - u_\infty^2\|_X &\leq \\ &\frac{1}{1 - \alpha C \max(K, 1)} \|g_1 - g_2\|_{L^2}. \end{aligned} \tag{8}$$

7 asymptotic expansion

Suppose $\tau = \tau_0$, $\varepsilon \rightarrow 0$. We construct an asymptotic expansion of u as $\varepsilon \rightarrow 0$.

Let us introduce the following notation:

$$u^{(k)} = \sum_{i=0}^k \varepsilon^i u_i(x, y).$$

Functions $u_i(x, y)$ are satisfying the following *linear* differential equations.

Assume

$$\left\{ \begin{array}{l} \frac{\partial u_0}{\partial x} - A \frac{\partial^2 u_0}{\partial y^2} = \alpha \left\{ c \frac{\partial}{\partial y} f(u_0) \right\}, \\ \frac{\partial u_i}{\partial x} - A \frac{\partial^2 u_i}{\partial y^2} - \alpha \left\{ c \frac{\partial}{\partial y} (f'(u_0) u_i) \right\} = \\ \alpha \left(c \sum_{p=2}^i \frac{\partial}{\partial y} \left(\frac{f^{(p)}(u_0)}{p!} \sum_{\substack{s_1 + \dots + s_p = i \\ 1 \leq s_1 \dots s_p \leq i}} u_{s_1} \dots u_{s_p} \right) \right. \\ \left. - \frac{1}{\tau} u_{i-1} + \frac{1}{\tau^2} \int_{-\infty}^y u_{i-1}(x, y') e^{\frac{y'-y}{\tau}} dy' \right), \\ u_i|_{x=0} = \delta_{i0} \phi(y), \quad i = 0, 1, 2, \dots \end{array} \right. \quad (9)$$

THEOREM 6. *The inequality holds:*

$$\|u^{(k)} - u\|_X \leq C \varepsilon^{k+1},$$

i.e. the asymptotic developpement $u^{(k)}$ converges to the exact solution u with respect to the norm $\|\cdot\|_X$ as $\varepsilon \rightarrow 0$ with the order ε^{k+1} .

8 Projects

We can consider problem (1), (2), (3) with the non-constant coefficients. If the parameters of the problem depend on some small variable δ , i.e. are fast oscillating, the equations describe some stratified heterogeneous media. In this case it's interesting to study the asymptotic behavior of the solutions as $\delta \rightarrow 0$ and try to obtain the corresponding homogenized problem.

9 References

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