A levelset method in shape optimization for variational inequations

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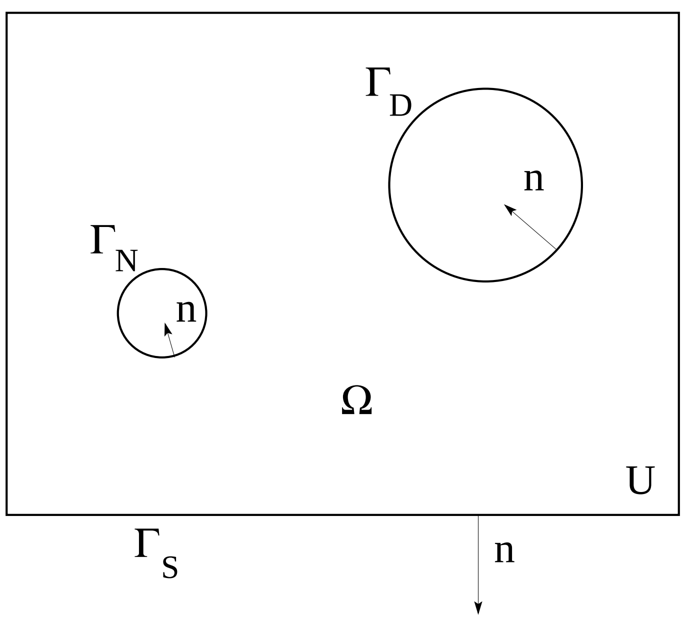
Introduction

Ω :	Domain in $\mathbb{R}^2$
n :	Outer normal vector to Ω
Γ_D :	Dirichlet conditions
Γ_N :	Neumann conditions
Γ_S :	Signiorini conditions or Dirichlet conditions



Linear problem

- Linear problem

$$\begin{cases} -\Delta u + u & = & f \quad \text{in} \quad \Omega \subset U \subset \mathbb{R}^2 \\ u & = & 0 \quad \text{on} \quad \Gamma_S \\ u & = & 0 \quad \text{on} \quad \Gamma_D, \\ \partial_n u & = & 0 \quad \text{on} \quad \Gamma_N, \end{cases}$$

Equivalently, u is solution of the variational equation

$$\int_\Omega \nabla u \cdot \nabla v + uv \, d\Omega = \int_\Omega f v d\Omega \quad \forall v \in H_0^1(\Omega)$$

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \cup \Gamma_S\}$$

Signiorini Problem

- Signiorini Problem

$$\begin{cases} -\Delta u + u &= f \quad \text{in} \quad \Omega \subset U \subset \mathbb{R}^2 \\ u \geq 0, \quad \partial_n u \geq 0, \quad u\partial_n u &= 0 \quad \text{on} \quad \Gamma_S \\ u &= 0 \quad \text{on} \quad \Gamma_D, \\ \partial_n u &= 0 \quad \text{on} \quad \Gamma_N, \end{cases}$$

Equivalently, u is the solution of a variational inequality

$$\int_\Omega \nabla u \cdot \nabla (v-u) d\Omega \geq \int_\Omega (f-u)(v-u) d\Omega \quad \forall v \in K$$

$$K = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D,\ v \geq 0 \text{ on } \Gamma_S\}$$

• We want to maximize the functionnal $J(\Omega)$ defined by

$$J(\Omega) = E(\Omega) + \lambda A(\Omega) - \mu P_c(\Omega)^2,$$

with $A(\Omega)$, $P_c(\Omega)$ and $E(\Omega)$ defined by

$$\begin{aligned} E(\Omega) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega fu \\ &= -\frac{1}{2}\int_\Omega fu, \\ A(\Omega) &= |\Omega|, \\ P_c(\Omega) &= \max(0, \mathcal{H}^1(\partial\Omega) - c), \end{aligned}$$

Existence of an optimal domain for the linear problem

Theorem 1. *(Bucur and Varchon)*

Let $\Omega_i = U \setminus \overline{\omega_i}$ *be a sequence of domains such that the number of connex components of* $\overline{\omega_i}$ *is uniformly bounded. If* $\overline{\omega_i}$ *converges to* $\overline{\omega}$ *in the sense of the Hausdorff metric, then the solution* u_i *of the Neumann problem*

$$\begin{cases} -\Delta u_i + u_i & = & f & \text{in } \Omega_i, \\ u_i & = & 0 & \text{on } \partial U. \\ \partial u_i / \partial n & = & 0 & \text{on } \partial \omega_i. \end{cases}$$

converges to the solution u *of the same problem defined on* $\Omega = U \setminus \overline{\omega}$ *in the following sense :*

$$u_i \overset{L^2(U)}{\longrightarrow} u \text{ and } \nabla u_i \overset{L^2(U;\mathbb{R}^2)}{\longrightarrow} \nabla u$$

if and only if $|\Omega_i| \to |\Omega|$. *(The functions are implicitly extended by* 0 *over* U*)*

Theorem 2. *The problem*

$$\max_{\Omega\in\mathcal{O}_k} J(\Omega)$$

admits at least one solution $\Omega \in \mathcal{O}_k = \{\Omega = U\setminus\overline{\omega},\ \omega\subset V\subset\subset U, \#\overline{\omega}\le k\}$.

Proof. Let Ω_i be a maximizing sequence such that $\Omega_i \xrightarrow{H^c} \Omega$. The number of connex components of Ω^c is bounded and

$$\mathcal{H}^1(\partial\Omega) \le \liminf_i \mathcal{H}^1(\partial\Omega_i).$$

$$\Omega_i \xrightarrow{H^c} \Omega \Longrightarrow \chi_{\Omega_i} \xrightarrow{L^1(U)} \chi_\Omega.$$

As a consequence, we get

$$|\Omega_i| \to |\Omega|.$$

Finally

$$\begin{aligned} |E(\Omega_i)-E(\Omega)| &= \left|\int_{\Omega_i} fu_i - \int_\Omega fu\right| \\ &= \left|\int_U fu_i\chi_{\Omega_i} - \int_U fu\chi_\Omega\right| \end{aligned}$$

$$\begin{aligned}|E(\Omega_i) - E(\Omega)| &= \left| \int_U f u_i(\chi_{\Omega_i} - \chi_\Omega) + \int_U f\chi_\Omega(u_i - u) \right| \\ &\leq \|u_i\|_2 \|f(\chi_{\Omega_i} - \chi_\Omega)\|_2 + \|f\chi_\Omega\|_2 \|u_i - u\|_2.\end{aligned}$$

According to theorem (1)

$$\|u_i\|_2 \to \|u\|_2 \qquad \text{and} \qquad \|u_i - u\|_2 \to 0$$

Finally

$$\|f(\chi_{\Omega_i} - \chi_\Omega)\|_2 \to 0 \Longrightarrow E(\Omega_i) \to E(\Omega),$$

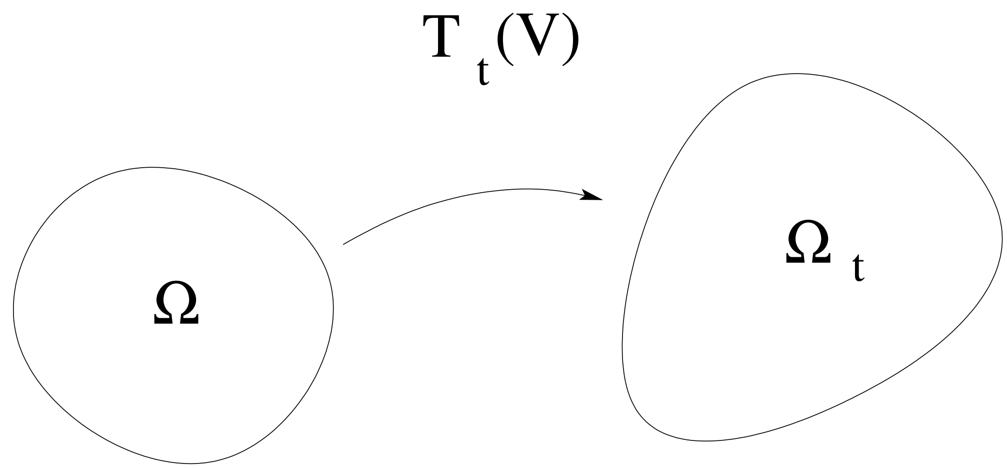
and

$$J(\Omega) \geq \limsup_i J(\Omega_i).$$

Thus J attains its maximum at Ω.

Shape derivative

Speed method



$V = V(x,t)$ smooth

$$\begin{cases} x'(t) & = & V(x(t),t), \quad 0 < t < \tau \\ x(0) & = & X \in U \subset \mathbb{R}^2 \end{cases}$$

$T_t(V)(X) = x(t)$ and $\Omega_t = T_t(V)(\Omega)$

- $dJ(\Omega; V) = \left.\dfrac{dJ(\Omega_t)}{dt}\right|_{t=0} = \lim_{t\to 0} \dfrac{J(\Omega_t) - J(\Omega)}{t}$
- $dJ(\Omega; V) > 0 \Longrightarrow J(\Omega_t) > J(\Omega), \quad t$ small.

Shape derivative for the linear problem

1) Dirichlet conditions on the boundary of the hole

$$J(\Omega) = E(\Omega) + \lambda A(\Omega) - \mu P_c(\Omega)^2 \quad \text{with} \quad \begin{cases} -\Delta u + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_S \\ u &= 0 \quad \text{on } \Gamma_D \end{cases}$$

$$dJ(\Omega; V) = \int_{\Gamma_D} \left(-\frac{1}{2}|\nabla u|^2 + \lambda - 2\mu P_c(\Omega)\mathcal{H} \right) V \cdot n \, d\sigma$$

Choice of V_n : we choose $V_n = V \cdot n$ so that $dJ(\Omega; V) > 0$.

$\longrightarrow$ • Dirichlet : $V_n = -\frac{1}{2}|\nabla u|^2 + \lambda - 2\mu P_c(\Omega)\mathcal{H}$ on Γ_D

2) Neumann conditions on the boundary of the hole

$$J(\Omega) = E(\Omega) + \lambda A(\Omega) - \mu P_c(\Omega)^2 \quad \text{with} \quad \begin{cases} -\Delta u + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_S \\ \partial_n u &= 0 \quad \text{on } \Gamma_N \end{cases}$$

$$dJ(\Omega; V) = \int_{\Gamma_N} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - uf + \lambda - 2\mu P_c(\Omega)\mathcal{H} \right) V \cdot n \, d\sigma$$

Choice of V_n : we choose $V_n = V \cdot n$ so that $dJ(\Omega; V) > 0$.

$\longrightarrow$ • Neumann : $V_n = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - uf + \lambda - 2\mu P_c(\Omega)\mathcal{H}$ on Γ_N

Shape derivative for the nonlinear problem

Consider transformation $F_\delta = I + \delta V$ with $\Omega_\delta = F_\delta(\Omega)$ and $u^\delta \in K_\delta$ solution of the following variational inequation

$$\int_{\Omega_\delta} \langle \nabla u^\delta, \nabla v - \nabla u^\delta \rangle \geq \int_{\Omega_\delta} (f - u^\delta)(v - u^\delta) \qquad \forall v \in K_\delta,$$

where

$$K_\delta = \left\{ v \in H^1(\Omega_\delta) \mid v \geq 0 \text{ on } \Gamma_S \, ; v = 0 \text{ on } \Gamma_D \right\} \ .$$

Energy functionnal

$$E(\Omega_\delta) = \frac{1}{2} \int_{\Omega_\delta} |\nabla u^\delta|^2 + (u^\delta)^2 - \int_{\Omega_\delta} f u^\delta.$$

The following limit

$$E'(\Omega, V) := \left. \frac{dE(\Omega_\delta)}{d\delta} \right|_{\delta=0} = \lim_{\delta \to 0^+} \frac{E(\Omega_\delta) - E(\Omega)}{\delta}$$

exists and is finite, linear and continuous with respect to the vector field V.

$$E(\Omega_\delta) = \frac{1}{2}\int_\Omega \left(\|(DF_\delta^T)^{-1}\nabla u_\delta\| + u_\delta^2\right) q_\delta \, dy - \int_\Omega f_\delta u_\delta q_\delta \, dy,$$

with

$$q_\delta = \det DF_\delta = 1 + \delta \mathsf{div}(V) + \delta^2 \det DV \ .$$

We define

$$f^\delta = q_\delta f_\delta.$$

The following limit exists

$$f' = \left.\frac{df^\delta}{d\delta}\right|_{\delta=0} = \lim_{\delta\to 0^+} \frac{f^\delta - f^0}{\delta} = \mathsf{div}(Vf) \in L^\infty(U) \ .$$

F_δ bijective between the convex sets K_0 and K_δ and

$$\|u_\delta - u\|_{H^1(\Omega)} \le C\delta$$

We introduce the notations

$$\begin{aligned} \pi(\Omega; u) &:= E(\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 \, dy - \int_\Omega f u \, dy, \\ \pi_\delta(\Omega; u_\delta) &:= E(\Omega_\delta) = \frac{1}{2}\int_\Omega \left(\|(DF_\delta^T)^{-1}\nabla u_\delta\| + u_\delta^2\right) q_\delta \, dy - \int_\Omega f_\delta u_\delta q_\delta \, dy. \end{aligned}$$

We get

$$\frac{E(\Omega_\delta) - E(\Omega)}{\delta} = \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u)}{\delta} \leq \frac{\pi_\delta(\Omega; u) - \pi(\Omega; u)}{\delta},$$

and passing to the limit we find

$$\begin{aligned} \limsup_{\delta \to 0^+} \frac{E(\Omega_\delta) - E(\Omega)}{\delta} &\leq \lim_{\delta \to 0^+} \frac{\pi_\delta(\Omega; u) - \pi(\Omega; u)}{\delta} \\ &\leq -\frac{1}{2} \int_\Omega \langle A\,(V)\, \nabla u, \nabla u \rangle - u^2 \mathrm{div}(V)\, dy - \int_\Omega g\,(V)\, u\, dy\ , \end{aligned}$$

with $A\,(V)$ and $g\,(V)$ defined by

$$A(V) = DV + DV^T - (\mathrm{div} V) I\ ,$$

$$g(V) = \mathrm{div}(fV)\ .$$

In the same way, we have another inequality

$$\frac{E(\Omega_\delta) - E(\Omega)}{\delta} = \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u)}{\delta} \geq \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u_\delta)}{\delta},$$

and passing to the limit, we obtain

$$\begin{aligned}
\liminf_{\delta \to 0^+} \frac{E(\Omega_\delta) - E(\Omega)}{\delta} &\geq \lim_{\delta \to 0^+} \frac{\pi_\delta(\Omega; u_\delta) - \pi(\Omega; u_\delta)}{\delta} \\
&\geq -\frac{1}{2} \int_\Omega \langle A(V) \nabla u, \nabla u \rangle - u^2 \mathsf{div}(V)\, dy - \int_\Omega g(V)\, u\, dy \ ,
\end{aligned}$$

thanks to the strong convergence of u_δ to u in $H^1(\Omega)$. Thus

$$\begin{aligned}
E'(\Omega, V) &:= \lim_{\delta \to 0} \frac{E(\Omega_\delta) - E(\Omega)}{\delta} \\
&= -\frac{1}{2} \int_\Omega \langle A(V) \nabla u, \nabla u \rangle - u^2 \mathsf{div}(V)\, dy - \int_\Omega g(V)\, u\, dy \ .
\end{aligned}$$

Topological derivative

The speed method does not allow topological changes (creation of a hole).
$\rightarrow$ *topological derivative (used by Nazarov, Sokolowski, Zochowski, Guillaume, Masmoudi, Amstutz, Feijoo, Novotny, Tarocco and many others).*

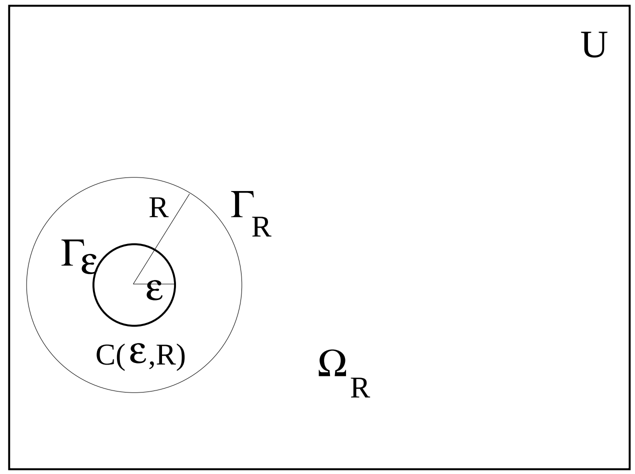
$$\omega_\varepsilon = B_{x_0}(\varepsilon), \quad \Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$$

$$J(\Omega_\varepsilon) = -\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + u_\varepsilon^2 + \lambda A(\Omega_\varepsilon) - \mu P_c(\Omega_\varepsilon)^2$$

Asymptotic expansion : $\quad J(\Omega_\varepsilon) = J(\Omega) + \rho(\varepsilon)\mathcal{T}_\Omega(x_0) + o(\rho(\varepsilon)),$
$\rho(\varepsilon) > 0, \quad \rho(\varepsilon) \to 0.$

If $\mathcal{T}_\Omega(x_0) > 0$ then $J(\Omega_\varepsilon) > J(\Omega)$.

Truncated problem



$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \quad \Omega_\varepsilon = U \setminus B_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon. \end{cases}$$

$$\begin{cases} -\Delta u_\varepsilon^R + u_\varepsilon^R = f & \text{in } \Omega_R, \quad \Omega_R = U \setminus B_R, \ R > \varepsilon, \\ u_\varepsilon^R = 0 & \text{on } \Gamma_S, \\ -\partial_n y_\varepsilon + \partial_n u_\varepsilon^R = A_\varepsilon(u_\varepsilon^R) & \text{on } \Gamma_R. \end{cases}$$

u_ε^R is solution of the following variational inequality

$$u_\varepsilon^R \in H^1_{\Gamma_S}(\Omega_R) : a_\varepsilon(u_\varepsilon^R, v) = l_\varepsilon(v) \qquad \forall v \in H^1_{\Gamma_S}(\Omega_R)$$

$$\begin{aligned} a_\varepsilon(u,v) &= \int_{\Omega_R} \langle \nabla u, \nabla v \rangle + uv\, dS + \int_{\Gamma_R} A_\varepsilon(u) v \\ l_\varepsilon(v) &= \int_{\Omega_R} fv - \int_{\Gamma_R} v \partial_n y_\varepsilon, \end{aligned}$$

Theorem 3. u_ε^R *verifies*

$$u_\varepsilon^R = u_{\varepsilon|\Omega_R},$$

The energy of the problem defined on Ω_ε is given by

$$\begin{aligned} E_\varepsilon(f) &= \frac{1}{2} \int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 + u_\varepsilon^2\, dS - \int_{\Omega_\varepsilon} f u_\varepsilon\, dS \\ &= -\frac{1}{2} \int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 + u_\varepsilon^2\, dS. \end{aligned}$$

Steklov-Poincare operator

The homogeneous problem

We introduce the homogeneous problem in the ring $C(\varepsilon, R) = B(R) \setminus \overline{B(\varepsilon)}$

$$\begin{cases} -\Delta w_\varepsilon + w_\varepsilon &= 0 \quad \text{in} \quad C(\varepsilon, R), \\ w_\varepsilon &= v \quad \text{on} \quad \Gamma_R. \\ w_\varepsilon &= 0 \quad \text{on} \quad \Gamma_\varepsilon. \end{cases}$$

The Steklov-Poincare operator is defined by

$$A_\varepsilon : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$$

$$A_\varepsilon(v) = \frac{\partial w_\varepsilon}{\partial n} \quad \text{on } \Gamma_R$$

Asymptotic expansion for A_ε

$$A_\varepsilon = A_0 + \frac{1}{|\ln \varepsilon|} B + o(\frac{1}{|\ln \varepsilon|^2}) \text{ in } \mathcal{L}(H^{1/2}(\Gamma_R); H^{-1/2}(\Gamma_R))$$

We define the energies

$$E^{(1)}(v) = \int_{B(R)} \|\nabla w\|^2 + w^2 dS, \qquad E_\varepsilon^{(1)}(v) = \int_{C(R,\varepsilon)} \|\nabla w_\varepsilon\|^2 + w_\varepsilon^2 dS,$$

Theorem 4. $E_\varepsilon^{(1)}(v)$ *has an expansion*

$$E_\varepsilon^{(1)}(v) = E^{(1)}(v) + \frac{\pi a_0(R)^2}{2I_0(R)^2|\ln\varepsilon|} + \mathfrak{R}(v),$$

with

$$|\mathfrak{R}(v)| \leq \frac{M}{|\ln\varepsilon|^2},$$

uniformly on bounded subsets of $H^{\frac{1}{2}}(\Gamma_R)$ *and*

$$v(R,\phi) = \frac{1}{2}a_0(R) + \sum_{k=1}^{\infty} a_k(R)\sin k\phi + b_k(R)\cos k\phi$$

$$I_k(r) = \sum_{m=0}^{\infty} \frac{\left(\frac{r}{2}\right)^{k+2m}}{m!(k+m)!}, \qquad k \geq 0$$

The non-homogeneous problem

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon &= f_{|C(R,\varepsilon)} & \text{in} & C(R,\varepsilon) \\ y_\varepsilon &= 0 & \text{on} & \Gamma_R \\ y_\varepsilon &= 0 & \text{on} & \Gamma_\varepsilon \end{cases}$$

with $f \in C^\infty(\mathbb{R}^2)$. We study the following function :

$$f_{|C(R,\varepsilon)} \mapsto \frac{\partial y_\varepsilon}{\partial n}_{|\Gamma_R} = g_\varepsilon.$$

Theorem 5. *g_ε admits the expansion*

$$g_\varepsilon = g_0 - \frac{h(R)}{2RI_0(R)\ln\varepsilon} + O((\ln\varepsilon)^{-2}),$$

with

$$h(R) = \frac{1}{\pi}\int_0^R \int_0^{2\pi} tf(t,\phi)\frac{I_0(t)K_0(R) - K_0(t)I_0(R)}{I_0(R)}\,dtd\phi.$$

We define the energies

$$E^{(2)}(f) = -\int_{B(R)} \|\nabla y\|^2 + y^2 \, dS, \qquad E_\varepsilon^{(2)}(f) = -\int_{C(R,\varepsilon)} \|\nabla y_\varepsilon\|^2 + y_\varepsilon^2 \, dS$$

Theorem 6. $E_\varepsilon^{(2)}(f)$ *has an expansion*

$$E_\varepsilon^{(2)}(f) = E^{(2)}(f) + \frac{\pi h(R)^2}{2|\ln \varepsilon|} + \mathfrak{R}(f),$$

with

$$|\mathfrak{R}(f)| \leq \frac{M}{|\ln \varepsilon|^2},$$

Theorem 7. *The energy of the problem defined on* Ω_ε *admits the expansion*

$$E_\varepsilon(f) = E(f) + \frac{\pi a_0(R)^2}{4I_0(R)^2|\ln\varepsilon|} + \frac{\pi h(R)^2}{4|\ln\varepsilon|} + \frac{\pi a_0 h(R)}{2I_0(R)|\ln\varepsilon|} + \mathfrak{R}(f)$$

with

$$|\mathfrak{R}(f)| \leq \frac{M}{|\ln\varepsilon|^2}$$

where the coefficient $a_0(R)$ *is defined by*

$$a_0(R) = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\,d\phi.$$

We have the following convergences

$$\lim_{R\to 0} h(R) = 0, \quad \lim_{R\to 0} a_0(R) = 2u(0), \quad \lim_{R\to 0} I_0(R) = 1$$

so that we can consider the following approximation for the topological derivative

$$\mathcal{T}_\Omega(0) = \frac{1}{4\pi}\left(\int_0^{2\pi} u(R,\phi)\,d\phi\right)^2$$

In fact, we can prove that

$$u(0) = \frac{a_0(R)}{2I_0(R)} + \frac{h(R)}{2}$$

As a consequence

$$E_\varepsilon(f) = E(f) + \frac{2\pi u(0)^2}{|\ln \varepsilon|} + \mathfrak{R}(f).$$

with

$$\mathfrak{R}(f) \leq \frac{M}{|\ln \varepsilon|^2}$$

- Dirichlet conditions on $\partial\omega_\varepsilon$

$$u_\varepsilon = 0 \text{ on } \ \partial\omega_\varepsilon$$

$$J(\Omega_\varepsilon) = J(\Omega) + \frac{\pi u(0)^2}{|\ln\varepsilon|} + o(1/\log\varepsilon) - 4\mu P_c(\Omega)\pi\varepsilon - \lambda\pi\varepsilon^2$$

$$\rightarrow \mathcal{T}_\Omega(x_0) = u_\Omega^2(x_0) \geq 0$$

- Neumann conditions on $\partial\omega_\varepsilon$

$$\partial_n u_\varepsilon = 0 \text{ on } \ \partial\omega_\varepsilon$$

$$J(\Omega_\varepsilon) = J(\Omega) + \pi\varepsilon^2\left(-|\nabla u(x_0)|^2 - \frac{1}{2}u(x_0)^2 + uf(x_0) - \lambda\right) + 4\mu P_c(\Omega)\pi\varepsilon + o(\varepsilon^2)$$

$$\rightarrow \begin{cases} \mathcal{T}_\Omega(x_0) = -|\nabla u(x_0)|^2 - \frac{1}{2}u(x_0)^2 + uf(x_0) - \lambda & \text{si } P_c(\Omega) = 0 \\ \mathcal{T}_\Omega(x_0) = 4\mu P_c(\Omega)\pi & \text{si } P_c(\Omega) > 0 \end{cases}$$

If $\mathcal{T}_\Omega(x_{max}) \geq \mathcal{T}_\Omega(x)$ for all $x \in \Omega$ in $\mathcal{T}_\Omega(x_{max}) > 0$, we drill a hole in Ω at x_{max}.

Signiorini problem in Ω_R

$$
\left\{
\begin{array}{rcll}
-\Delta u_\varepsilon^R + u_\varepsilon^R & = & f & \text{in} \quad \Omega_R, \quad \Omega_R = U \setminus B_R,\ R > \varepsilon, \\
\partial_n u_\varepsilon^R & = & 0 & \text{on} \quad \Gamma_N, \\
u_\varepsilon^R \geq 0,\ \partial_n u_\varepsilon^R \geq 0, u_\varepsilon^R \partial_n u_\varepsilon^R & = & 0 & \text{on} \quad \Gamma_S \\
-\partial_n y_\varepsilon + \partial_n u_\varepsilon^R & = & A_\varepsilon(u_\varepsilon^R) & \text{on} \quad \Gamma_R,
\end{array}
\right.
$$

verifies $u_\varepsilon^R = u_R + \varepsilon^2 q_R + o(\varepsilon^2)$ in $H^1(\Omega_R)$. We also have

$$
u_\varepsilon^R = u_\varepsilon|_{\Omega_R}\ \forall \varepsilon \geq 0
$$
$$
q_R = q|_{\Omega_R}
$$

The outer topological derivative q of the solution u is given by

$$\int_\Omega \nabla q \nabla (v-q) + q(v-q) + \int_{\Gamma_R} B(u)(v-q) \geq 0$$

where B is the derivative of the Steklov-Poincare operator and

$$q \in S(u) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D,\ v \geq 0 \text{ on } \Xi(u), a(u,v) = 0\}$$

$$a(u,v) = \int_\Omega \nabla u \nabla v + uv - \int_\Omega fv$$

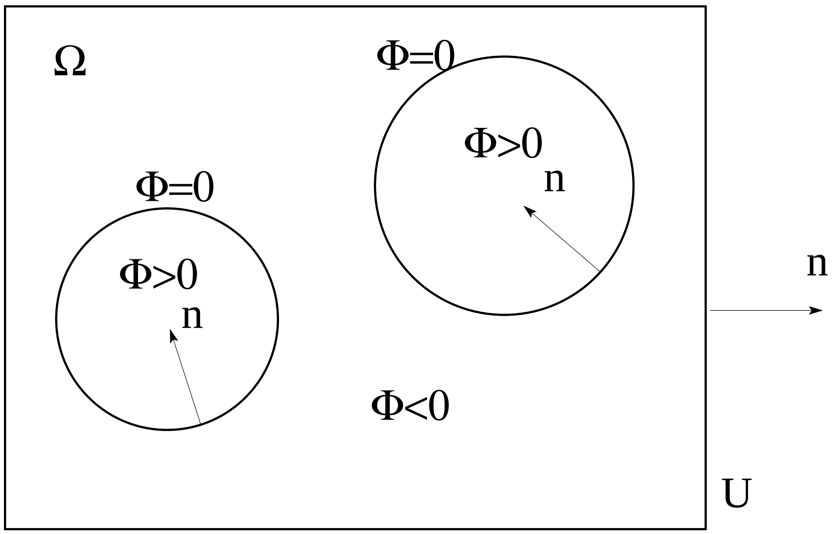
$$\Xi(u) = \{x \in \Gamma_s \mid u(x) = 0\}$$

We also obtain an expression similar to the linear case for the topological derivative of the functional J.

The levelset function

The levelset function $\phi = \phi(x,t)$ is defined for all $x \in U$

$$\Omega = \{x \in U,\ \phi(x,t) < 0\}$$
$$\Gamma = \{x \in U,\ \phi(x,t) = 0\}$$



$$\left.\begin{array}{l} \phi_t + V \cdot \nabla\phi = 0 \text{ in } U \\ n = \dfrac{\nabla\phi}{|\nabla\phi|} \end{array}\right\}$$

$\Rightarrow$ Hamilton-Jacobi equation
$\phi_t + V_n|\nabla\phi| = 0$ in U

- Dirichlet : $V_n = -\frac{1}{2}|\nabla u|^2 + \lambda - 2\mu P_c(\Omega)\mathcal{H}$ on Γ_D
- Neumann : $V_n = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - uf + \lambda - 2\mu P_c(\Omega)\mathcal{H}$ on Γ_N

Remarks : V_n must be defined everywhere in U, not only on $\Gamma \rightarrow$ extension
We want to have $|\nabla\phi| \simeq 1$.

Extension of the normal speed

We calculate V_{ext} in U so that $V_{ext} = V_n$ on $\Gamma = \Gamma_D$ or Γ_N and

$$\nabla V_{ext} \cdot \nabla \phi = 0 \text{ in } U.$$

Then we get $|\nabla \phi| = 1$ for all t.

Construction de V_{ext}

Solve (for large T)

$$\begin{aligned} & q_\tau + S(\phi)\frac{\nabla \phi}{|\nabla \phi|} \cdot \nabla q = 0, && \text{in } U \times (0, T) \\ & q(x, \tau = 0) = p(x), && x \in U \end{aligned}$$

where $p = V_n$ on Γ and $p = 0$ otherwise.

S is close to the sign function

$$S(d) = \frac{d}{\sqrt{d^2 + |\nabla d|^2 \epsilon^2}}, \;\; \epsilon = \min(\Delta x, \Delta y).$$

Algorithm

- Initialisation of ϕ^0, initial domain Ω_0.
- For k from 0 to K

(1) Calculation of the topological derivative in Ω_k

i. calculation of u_{Ω_k} in Ω_k

ii. calculation of the topological derivative

iii. a hole is created $\rightarrow$ domain Ω_k^1

(2) Solve Hamilton-Jacobi equation in U

* For p from 1 to P

i. calculation of $u_{\Omega_k^p}$ in Ω_k^p

ii. calculation of V_n^p on Γ_k^p and its extension to U

iii. solve Hamilton-Jacobi equation : $\phi_t + V_n^p |\nabla \phi| = 0$, $\rightarrow$ domain Ω_k^{p+1}

end for p

* $\rightarrow \phi^{k+1}$ in D and new domain $\Omega_{k+1} = \Omega_k^P$

- end for k

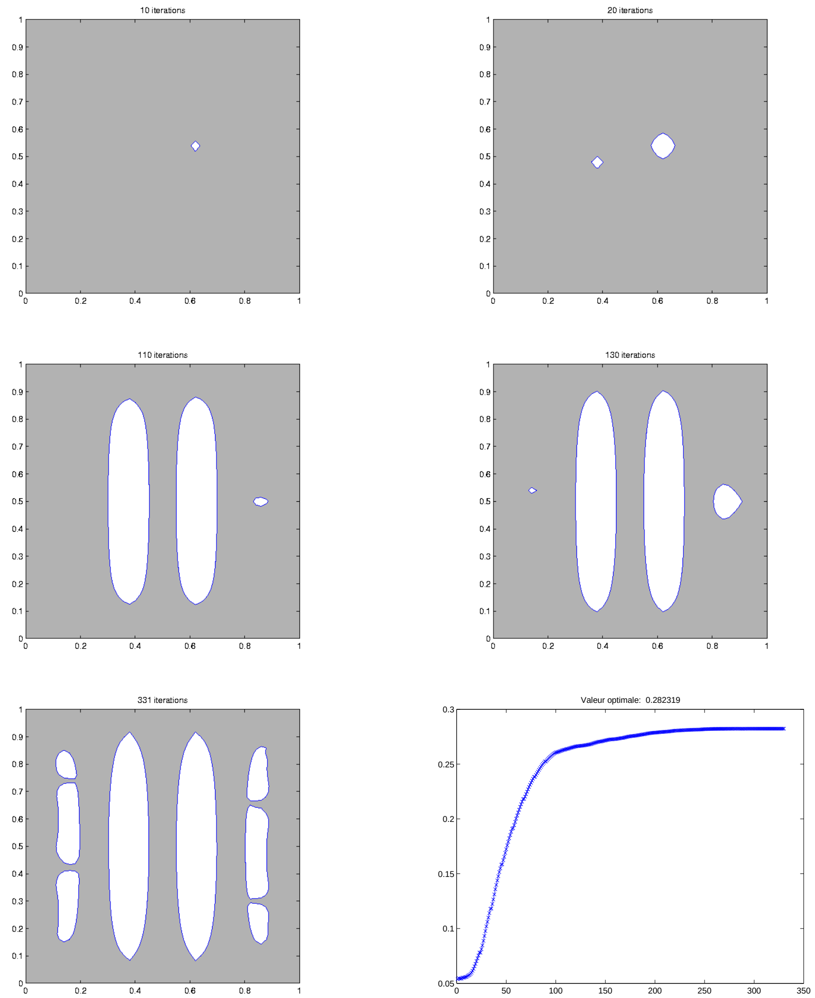


FIG. 1. The initial domain is full, we use the topological derivative
$f = 10\sin^2(4\pi x)$, $\lambda = 0.5$, $\mu = 0$
Optimal value : $J(\Omega_1) = 0,282319$

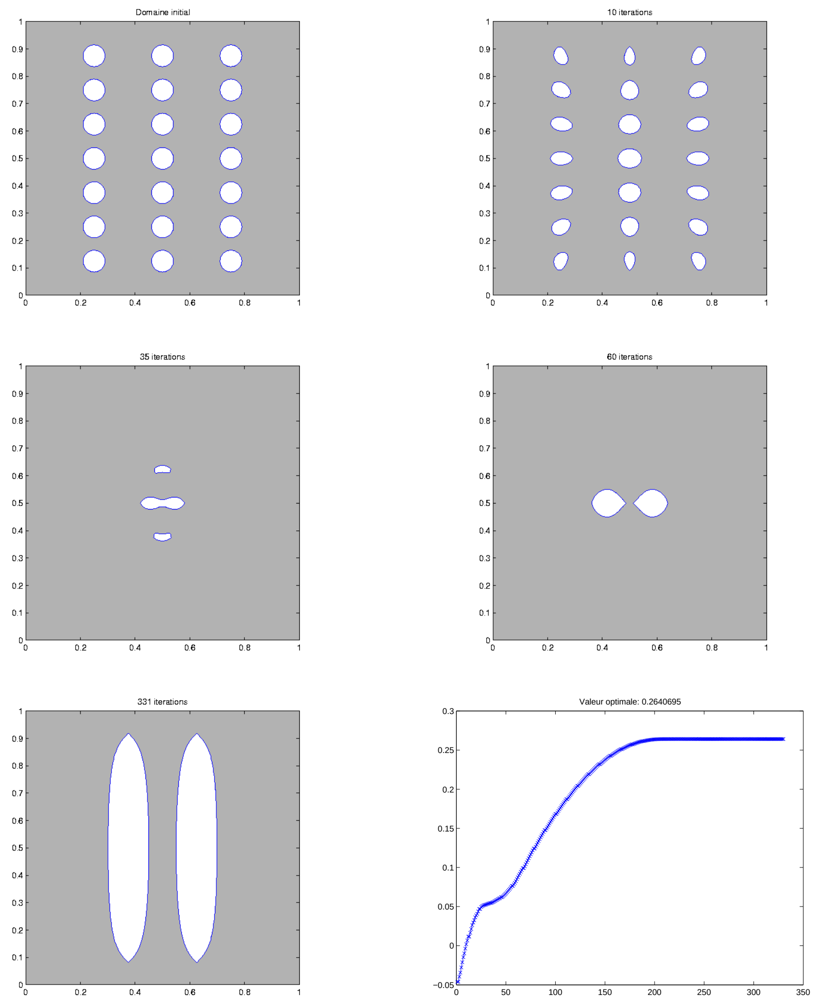


FIG. 1. 21 holes in the initial domain, no topological derivative
$f = 10\sin^2(4\pi x)$, $\lambda = 0.5$, $\mu = 0$
Optimal value : $J(\Omega_2) = 0,2640695$

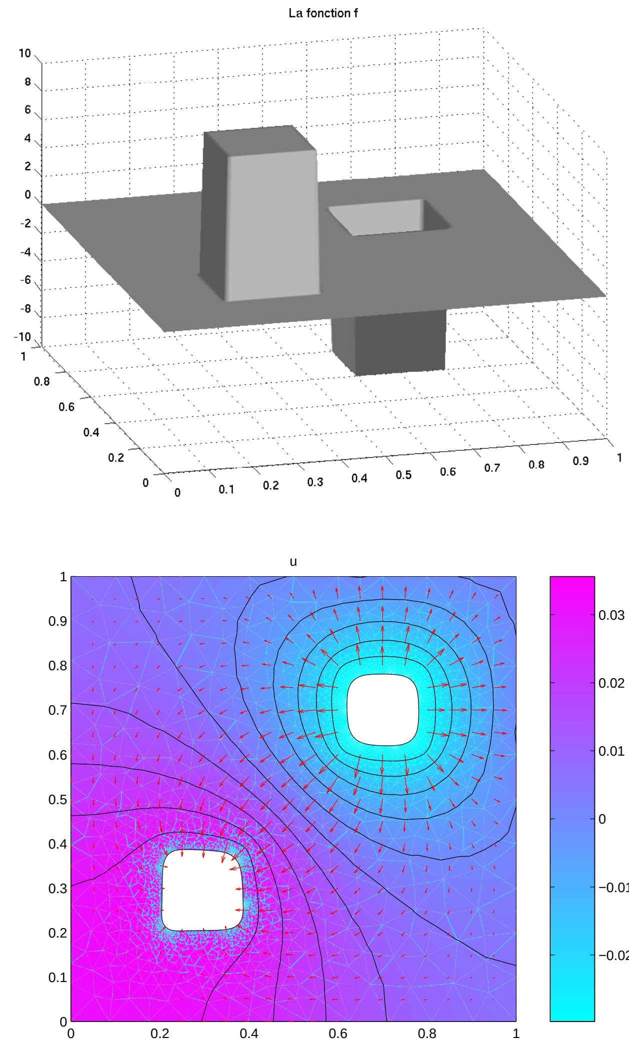


FIG. 1. Function f and solution u in the final domain

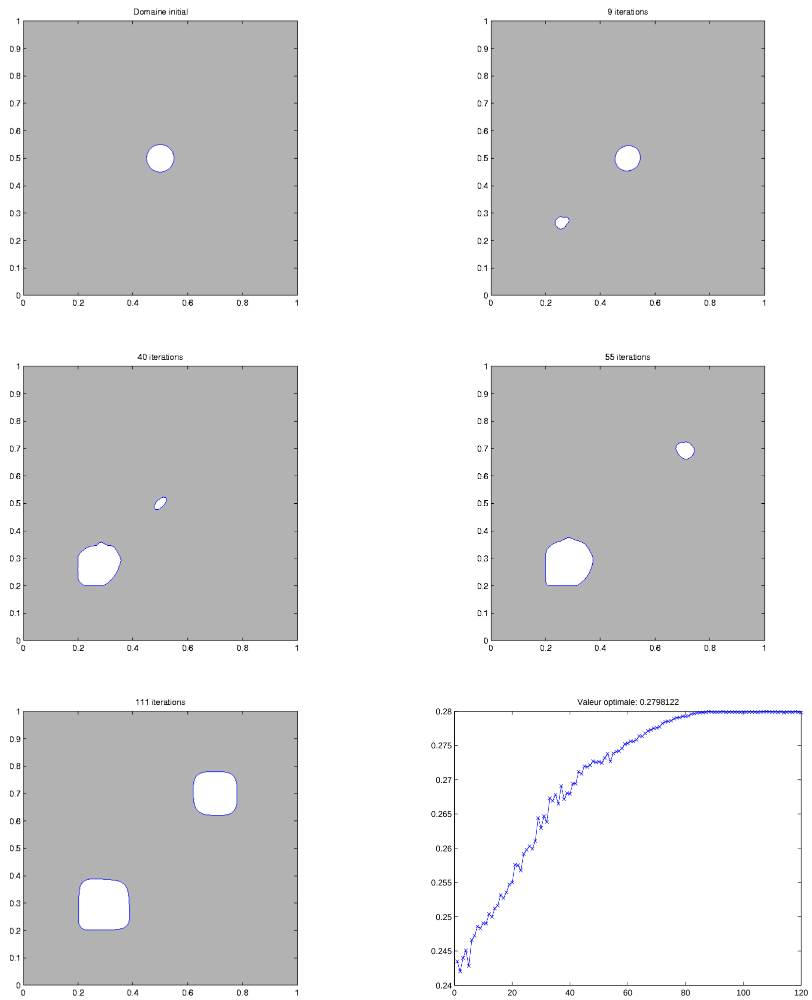


FIG. 1. The non-linear case
$\lambda = 0.3$, $\mu = 0.001$, $c = 0.6$

Conclusion

• Without topological derivative, the algorithm is very sensitive to the initial domain, a good initial guess is necessary.

• With topological derivative, we can start with any initial domain, in particular the full domain.

• Heavy calculations without topological derivative, due to an initial domain with a lot of holes.

• For the non-linear problem, it is possible to define a shape derivative and a topological derivative, and we obtain the same formulas as for the linear problem.