An energy conserving approximation for elastodynamic contact problems

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Plan

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- Classical time integration schemes
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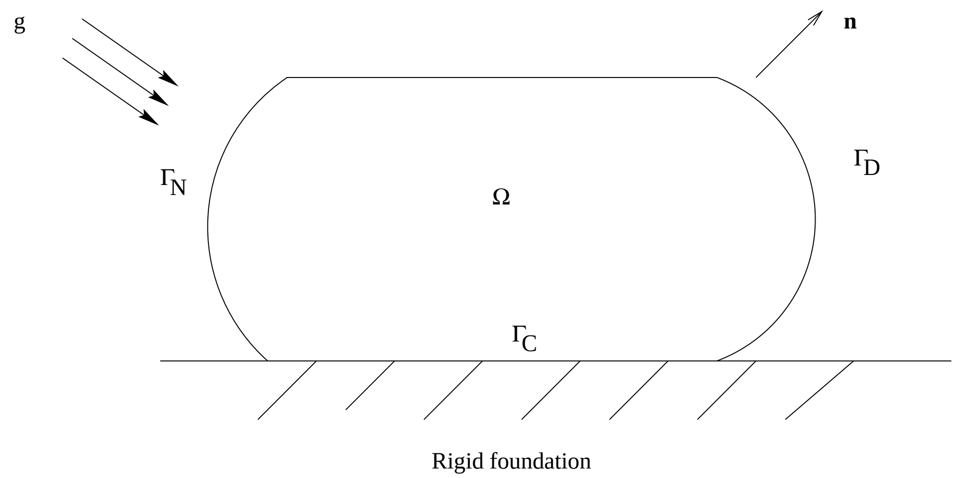
Contact problem in elastodynamics



The elastodynamic contact problem reads as : find $u : [0,T] \longrightarrow \Omega$ such that

$$\begin{aligned} \rho\ddot{u} - div\sigma(u) = f, &\quad \text{in }]0,T] \times \Omega, \\ \sigma(u) = \mathcal{A}\varepsilon(u), &\quad \text{in }]0,T] \times \Omega, \\ \sigma(u)n = g, &\quad \text{on }]0,T] \times \Gamma_N, \\ u = 0, &\quad \text{on }]0,T] \times \Gamma_D, \\ u(0) = u^0, \dot{u}(0) = v^0, &\quad \text{in } \Omega. \end{aligned}$$

Contact problem in elastodynamics

Assuming that Γ_C has a $\mathcal{C}^1$ regularity, we have :

$$u_N = u.n, \quad u_T = u - u_N n,$$

$$\sigma_N(u) = (\sigma(u)n).n, \quad \sigma_T(u) = \sigma(u)n - \sigma_N n.$$

Then, the unilateral contact frictionless condition is expressed as follows :

$$u_N \leq 0, \sigma_N \leq 0, u_N \sigma_N = 0 \text{ and } \sigma_T = 0, \text{ on }]0,T] \times \Gamma_C.$$

- Many works have been devoted to the numerical solution of contact problem in elastodynamics.
- Schatzman-Paoli (2001), Laursen-Chawla (1997, 2002)...
- For purely contact elastodynamic problems (hyperbolic problems), existence result has been proved in a scalar two dimensional case by Lebeau-Schatzman (1984), Kim (1989) and in the vector case with a modified contact law by Renard-Paumier (2003).

Nodal discretization

We consider a Lagrange FEM. Let $a_1, ..., a_d$ the finite element nodes, $\varphi_1, ..., \varphi_d$ be the basis functions and $I_C = \{i : a_i \in \Gamma_C\}$. Let U be the vector of degrees of freedom of the FEM displacement $u_h(x)$:

$$u_h(x) = \sum_{1 \leq i \leq d} u_i \varphi_i \quad \text{and} \quad U = (u_i) \in \mathbb{R}^d.$$

Then, the space semi-discretized elastodynamic contact problem reads as follows.

Find $U : [0, T] \longrightarrow \mathbb{R}^d$ such that

$$\begin{cases} M_0 \ddot{U} + KU = L + \sum_{i \in I_C} \lambda_N^i N_i, \\ \lambda_N^i \leq 0,\ U.N_i \leq 0,\ \lambda_N^i (U.N_i) = 0, \forall\, i \in I_C, \\ U(0) = U^0, \dot{U}(0) = V^0, \end{cases} \tag{1}$$

where

$$K = (K_{ij}) \text{ such that } K_{ij} = \int_\Omega \mathcal{A}\,\varepsilon(\varphi_i) : \varepsilon(\varphi_j)\,dx,$$

$$L = (L_i) \text{ such that } L_i = \int_\Omega f.\varphi_i dx + \int_{\Gamma_C} g.\varphi_i dx,$$

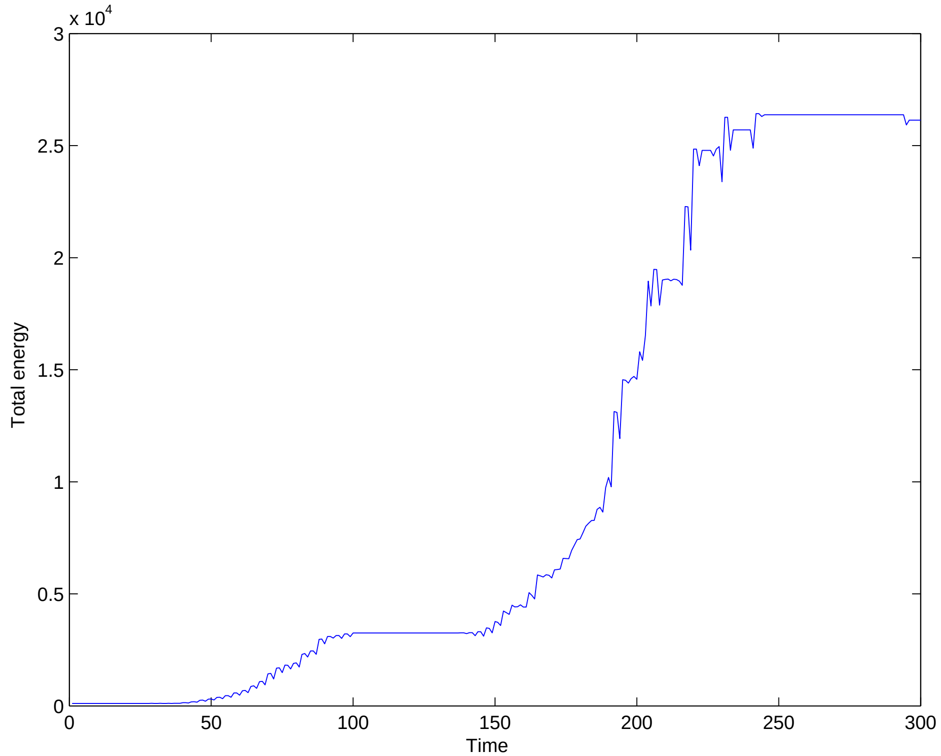
$$M_0 = ((M_0)_{ij}) \text{ such that } (M_0)_{ij} = \int_\Omega \rho\,\varphi_i.\varphi_j\,dx.$$

The normal displacement at the node a_i on the contact surface is equal to : $u^h_N(a_i) = N_i^T U$.
The multipliers λ^i_N defi ne the nodal contact forces vector : $\Lambda_N = (\lambda^i_N)$.

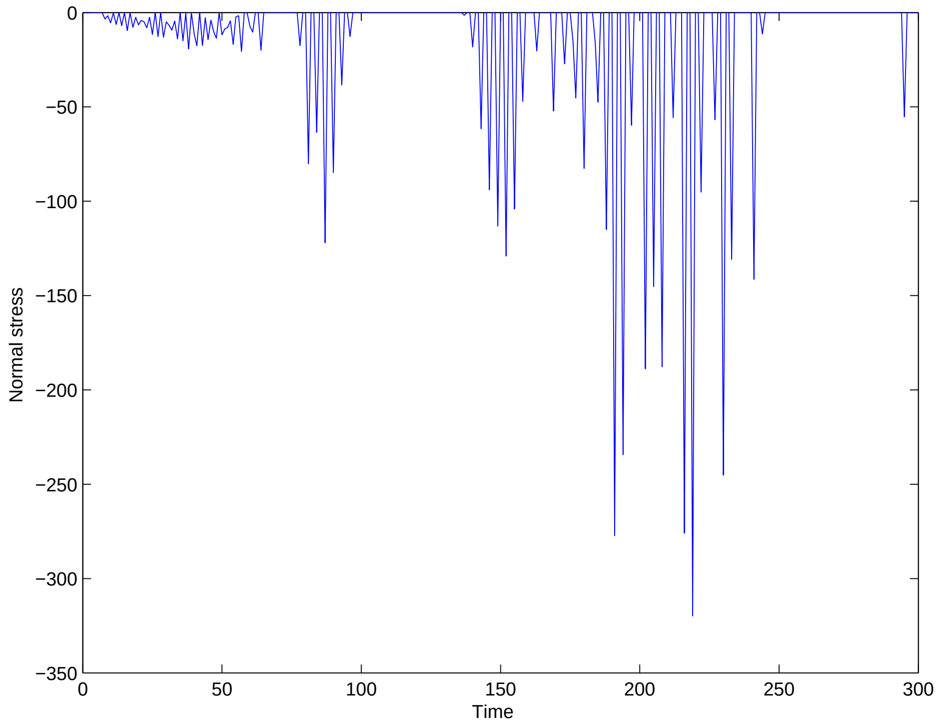
Remark :

- Problem (1) is ill-posed (Moreau 1983, 1986)
- Uniqueness can be recovered, for rigid bodies, by introducing an impact law with a restitution coeffi cient (Moreau 1999, Paoli-Schatzman 1993)
- This seems not to be completly satisfying for deformable bodies because whatever the restitution coeffi cient value is, the system tends to a global restitution of energy when the mesh parameter goes to zero.

The θ-method $\theta = 0.5$:



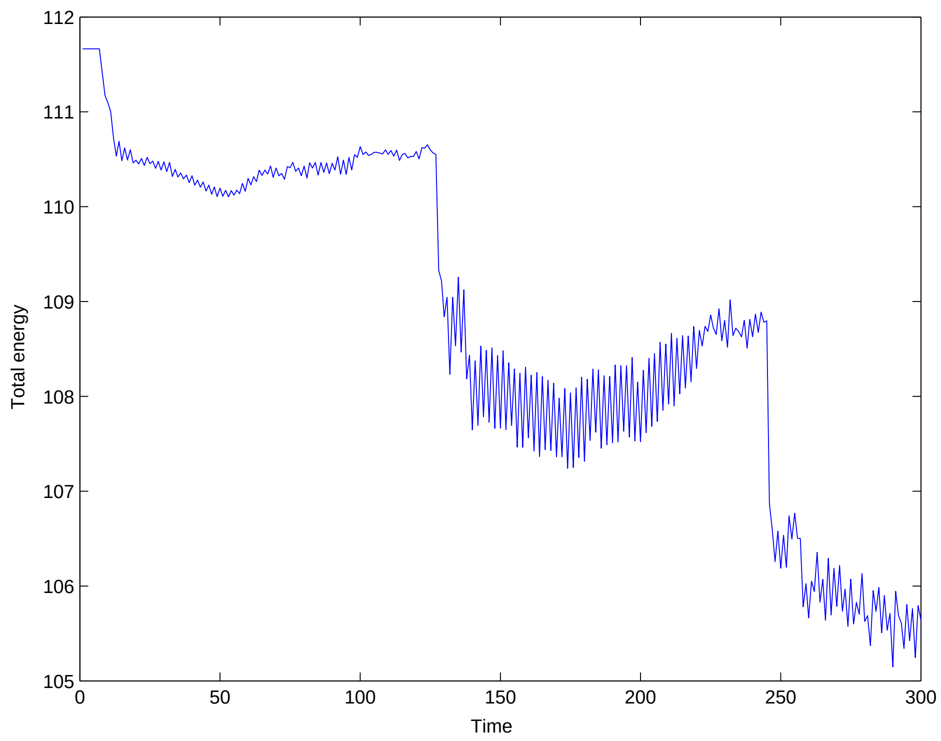
Energy

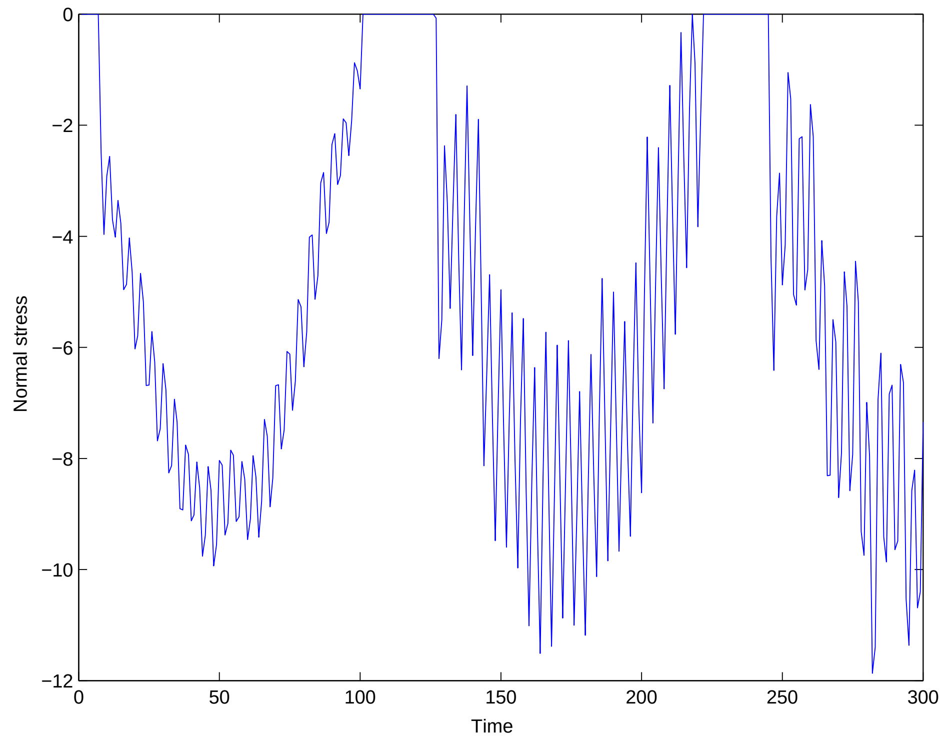


Normal stress

FIG. 1: *energy and normal stress evolution for the θ-method.*

The Newmark scheme $\beta = \gamma = 0.5$:



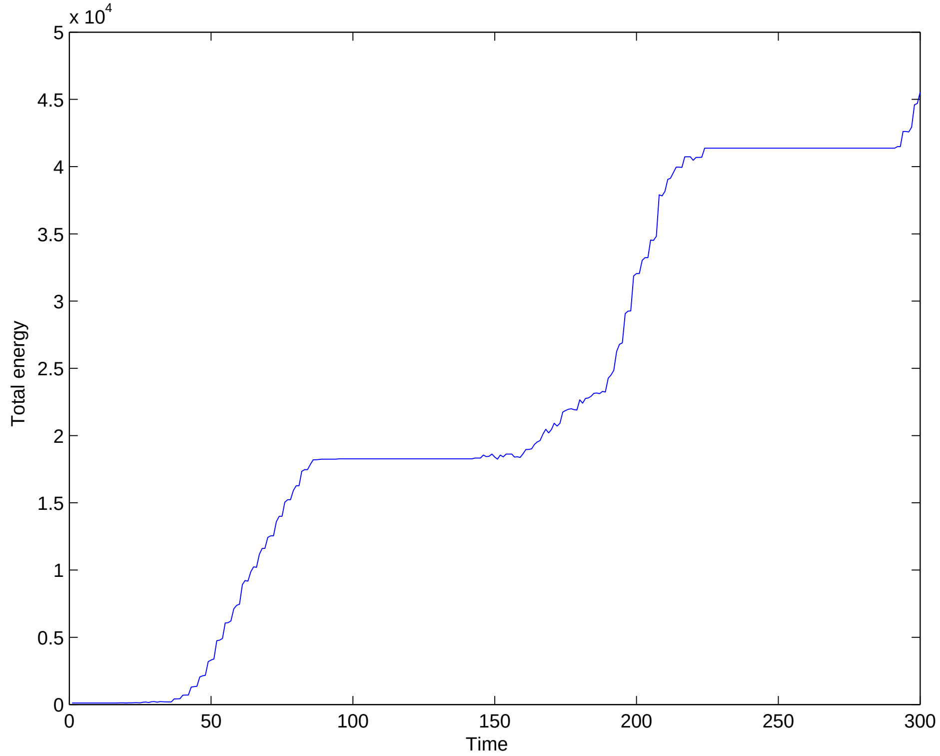


Energy

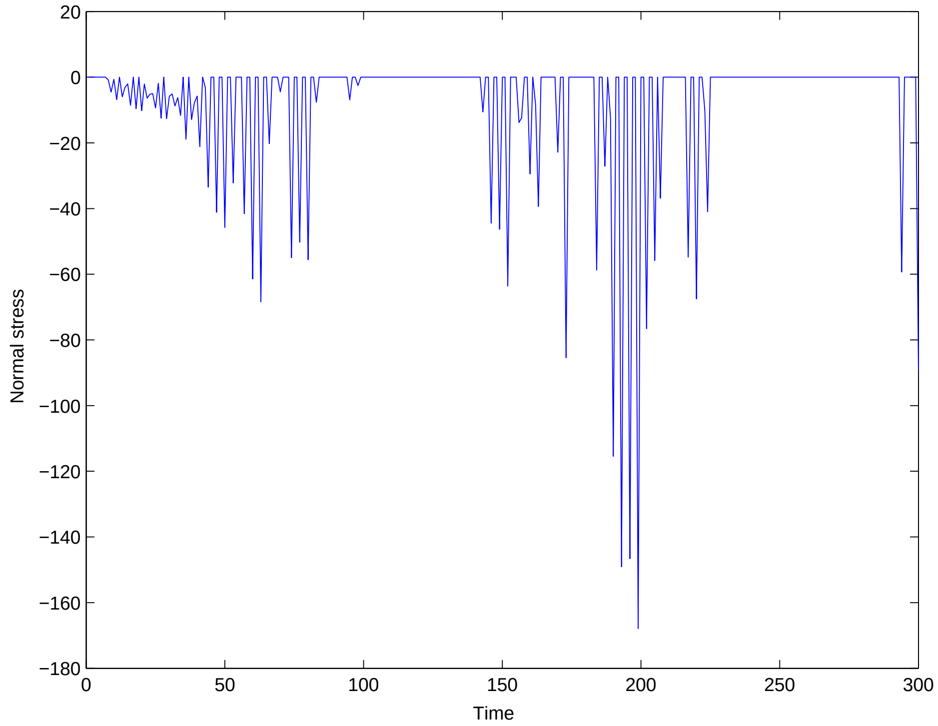
Normal stress

FIG. 2: *energy and normal stress evolution for the Newmark scheme.*

The midpoint scheme :



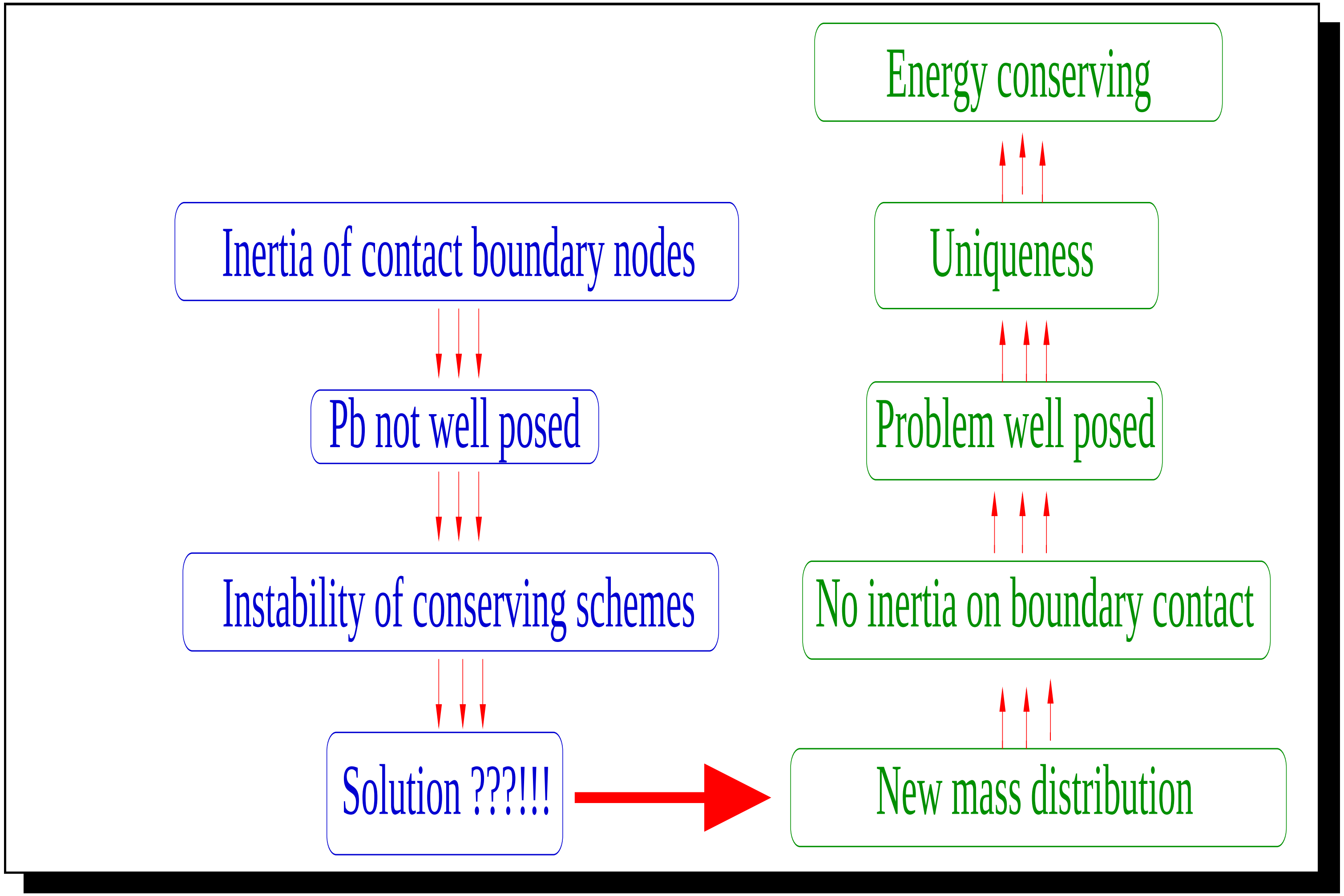
Energy



Normal stress

FIG. 3: *energy and normal stress evolution for the midpoint scheme.*

Redistribution Mass Method



We denote M the modifi ed mass matrix such that :

$$\inf_{\left\{\begin{array}{l} {N_i}^T M N_j = 0,\ \forall\, i, j \in I_C, \\ \text{the same total mass,} \\ \text{the same gravity center,} \\ \text{the same momentum inertia.} \end{array}\right.} \frac{1}{2}\|M - M_0\|^2 \qquad (2)$$

In addition, M must be symmetric and positive semi-defi nite like M_0.

Theorem 1

Let us suppose that the load vector $L(t)$ is a Lipschitz continuous function on $[0, T]$. Then, there exists one and only one Lipschitz continuous function (U, Λ_N) satisfying the finite element elastodynamic system with contact (1) at each time $t \in [0, T]$.

Proof. If we numerate the degrees of freedom such that the last ones are the nodes on the contact boundary, we can split each matrix and vector in an interior part and a contact boundary one as follows :

$$M = \begin{pmatrix} \overline{M} & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \overline{K} & C^T \\ C & \widetilde{K} \end{pmatrix}, \quad U = \begin{pmatrix} \overline{U} \\ \widetilde{U} \end{pmatrix},$$

$$L = \begin{pmatrix} \overline{L} \\ \widetilde{L} \end{pmatrix} \text{ and } \quad N_i = \begin{pmatrix} 0 \\ \widetilde{N}_i \end{pmatrix}.$$

Then, the system (1) is written :

$$\left\{\begin{array}{l} \overline{M}\,\ddot{\overline{U}} + \overline{K}\,\overline{U} = \overline{L} - C^T\widetilde{U}, \\ \\ C\overline{U} + \widetilde{K}\widetilde{U} = \widetilde{L} + \sum_{i\in I_C} \lambda_N^i \widetilde{N}_i, \\ \\ \widetilde{N}_i^T\widetilde{U} \leq 0,\ \ \lambda_N^i \leq 0,\ \ \lambda_N^i(\widetilde{N}_i^T\widetilde{U}) = 0 \quad \forall\, i \in I_C. \end{array}\right. \tag{3}$$

The following sub-system of (3) :

$$\left\{\begin{array}{l} \widetilde{K}\widetilde{U} + C\overline{U} = \widetilde{L} + \sum_{i\in I_C} \lambda_N^i \widetilde{N}_i, \\ \\ \widetilde{N}_i^T\widetilde{U} \leq 0,\ \ \lambda_N^i \leq 0,\ \ \lambda_N^i(\widetilde{N}_i^T\widetilde{U}) = 0 \quad \forall\, i \in I_C \end{array}\right. \tag{4}$$

can be expressed as follows :

$$a(\widetilde{U}, \widetilde{V} - \widetilde{U}) \geq l_{\overline{U}}(\widetilde{V} - \widetilde{U}) \quad \forall\, \widetilde{V} \in Q, \tag{5}$$

where

$a(\widetilde{U},\widetilde{V}) = \widetilde{V}^T\widetilde{K}\widetilde{U},\ l_{\overline{U}}(\widetilde{V}) = \widetilde{V}^T\widetilde{L} - \widetilde{V}^T C\overline{U}$ and $Q = \{V : \widetilde{N}_i^T\widetilde{V} \leq 0,\ i \in I_C\}$.

- $\widetilde{U}$ is uniquely defi ned from the variational inequality (5) for a given $\overline{U}$.
- $\widetilde{U}$ is a Lipschitz continuous function of $\overline{U}$.
- The fi rst equation in the system (3) is a second order Lipschitz ODE ($\overline{U}$).
- This equation has a unique solution $\overline{U}$ with a Lipschitz continuous derivative.
- There also exists one and only one Lipschitz continuous function in time $\widetilde{U}$ solution to the variational inequality (5).
- The second equation in the system (1) allows us to defi ne a unique Lipschitz continuous function $\lambda_N^i\ (i \in I_C)$ in time.

Proposition 1

The solution (U, Λ_N) to the space semi-discretized problem (1) satisfi es the following condition at each node on Γ_C :

$$\lambda_N^i (N_i^T \dot{U}) = 0 \ \text{ a.e. on } (0,T) \quad (i \in I_C). \tag{6}$$

Remark Formulation (6) is the so-called persistence contact condition in elastodynamics.

Proof. Using system (1) and the complementary conditions on the contact area, we have :

$$\lambda_N^i = 0 \ \text{ on } \ \text{Supp}(N_i^T U) = \omega_i \subset [0,T] \quad (i \in \Gamma_C).$$

The continuity of λ_N^i on $[0,T]$ implies that : $\lambda_N^i = 0$ on $\overline{\omega_i}$.

On the other hand, $N_i^T \dot{U} = 0$ a.e. on θ_i, where θ_i is the complementary part in $[0,T]$ of the interior of ω_i. Hence $\lambda_N^i (N_i^T \dot{U}) = 0$, a.e. on $(0,T)$.

Theorem 2

Assuming that the load vector L is constant in time, the fi nite element elastodynamic system with unilateral contact (1) is energy conserving.

Proof. The discrete energy of system (1) is given by :

$$E(t) = J(U, \dot{U}) = \frac{1}{2}\dot{U}^T M \dot{U} + \frac{1}{2} U^T K U - U^T L.$$

The fi rst equation in (1) implies :

$$\frac{1}{2}\dot{U}^T M \ddot{U} + \frac{1}{2}\dot{U}^T K U = \dot{U}^T L + \sum_{i \in I_C} \lambda_N^i \dot{U}^T N_i.$$

Integrating from 0 to t, it follows :

$$\frac{1}{2}\dot{U}^T M \dot{U} + \frac{1}{2} U^T K U - U^T L = \sum_{i \in I_C} \int_0^t \lambda_N^i \, \dot{U}^T N_i \, ds + E(0).$$

In other words, one has :

$$E(t) = \sum_{i \in I_C} \int_0^t \lambda_N^i \, \dot{U}^T N_i \, dt + E(0) \quad \forall t \in [0, T].$$

Thanks to Proposition 1, we finally obtain :

$$E(t) = E(0) \quad \forall t \in [0, T].$$

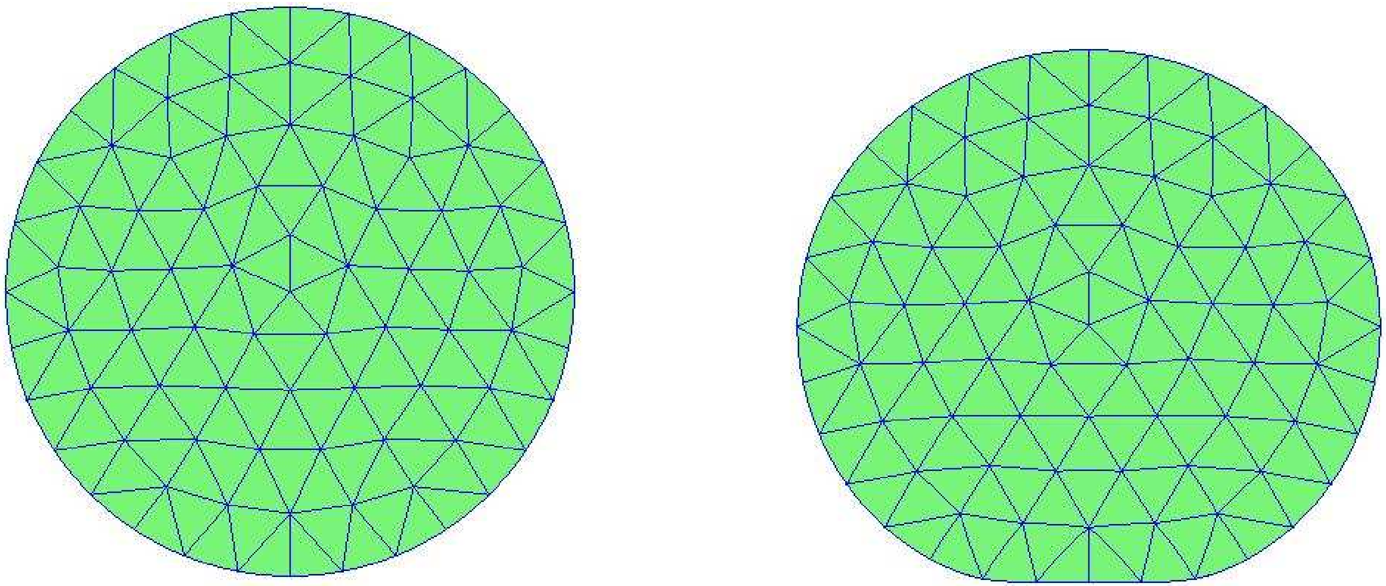


FIG. 4: *a disc before and during the first contact.*

We denote A the lowest point of the disc (the first point being in contact).

Disc and the resolution method properties	Values
ρ, diameter	$6\ 10^{-6} kg/cm^3$, $20\ cm$
Lamé coefficients	$\lambda = 10\ GP$, $\mu = 5\ GP$
u^0, v^0	$1\ cm$, $-100\ cm/s$
Time parameter	$10^{-3} s$
Simulation time	$0.3\ s$
Mesh parameter	$\simeq 2\ cm$

TAB. 1: *characteristics of the elastic disc and of the resolution method*

With the standard mass matrix With the redistributed mass matrix

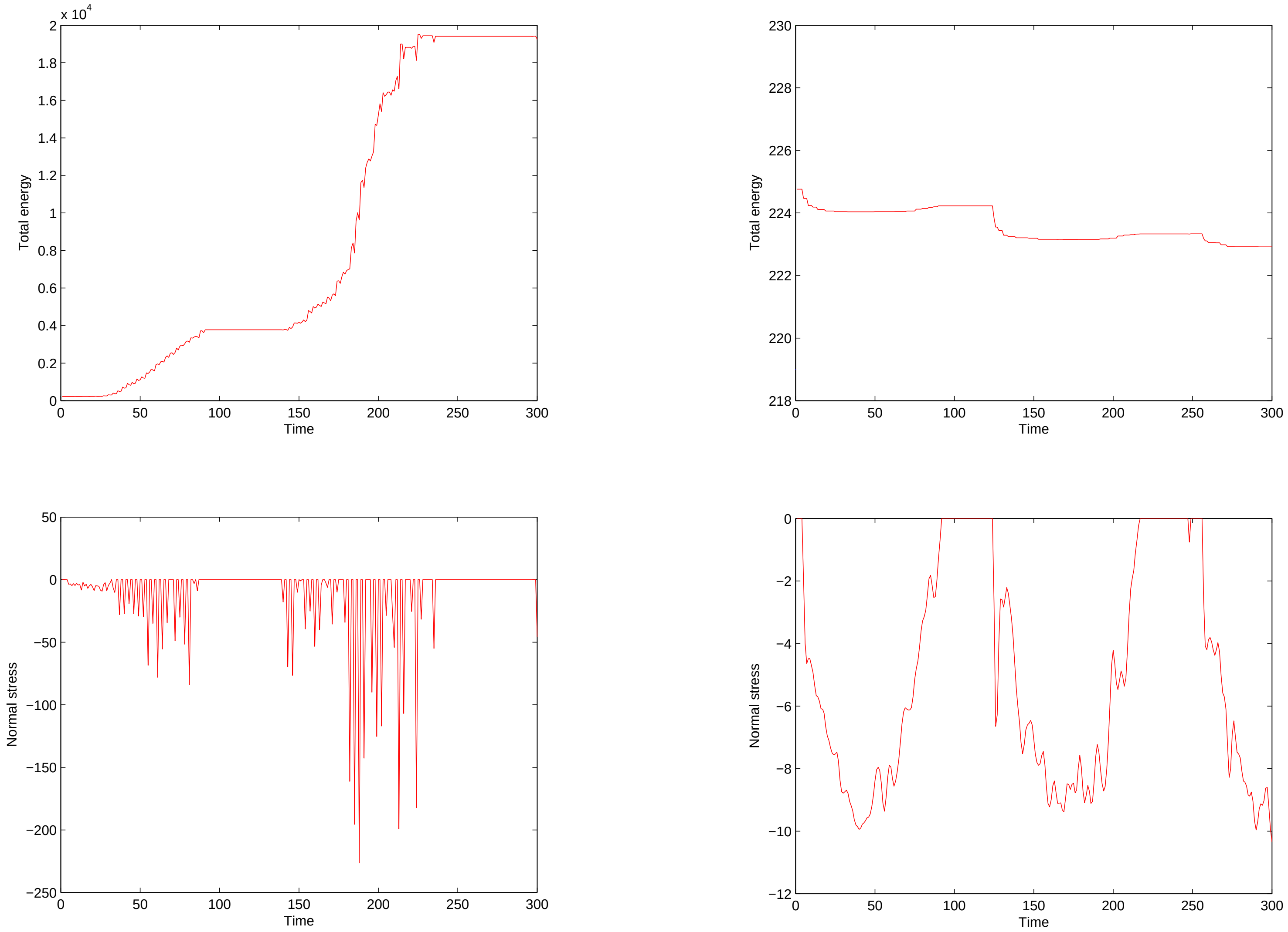


FIG. 5: *energy and normal stress evolution for the Crank Nicholson scheme.*

With the standard mass matrix With the redistributed mass matrix

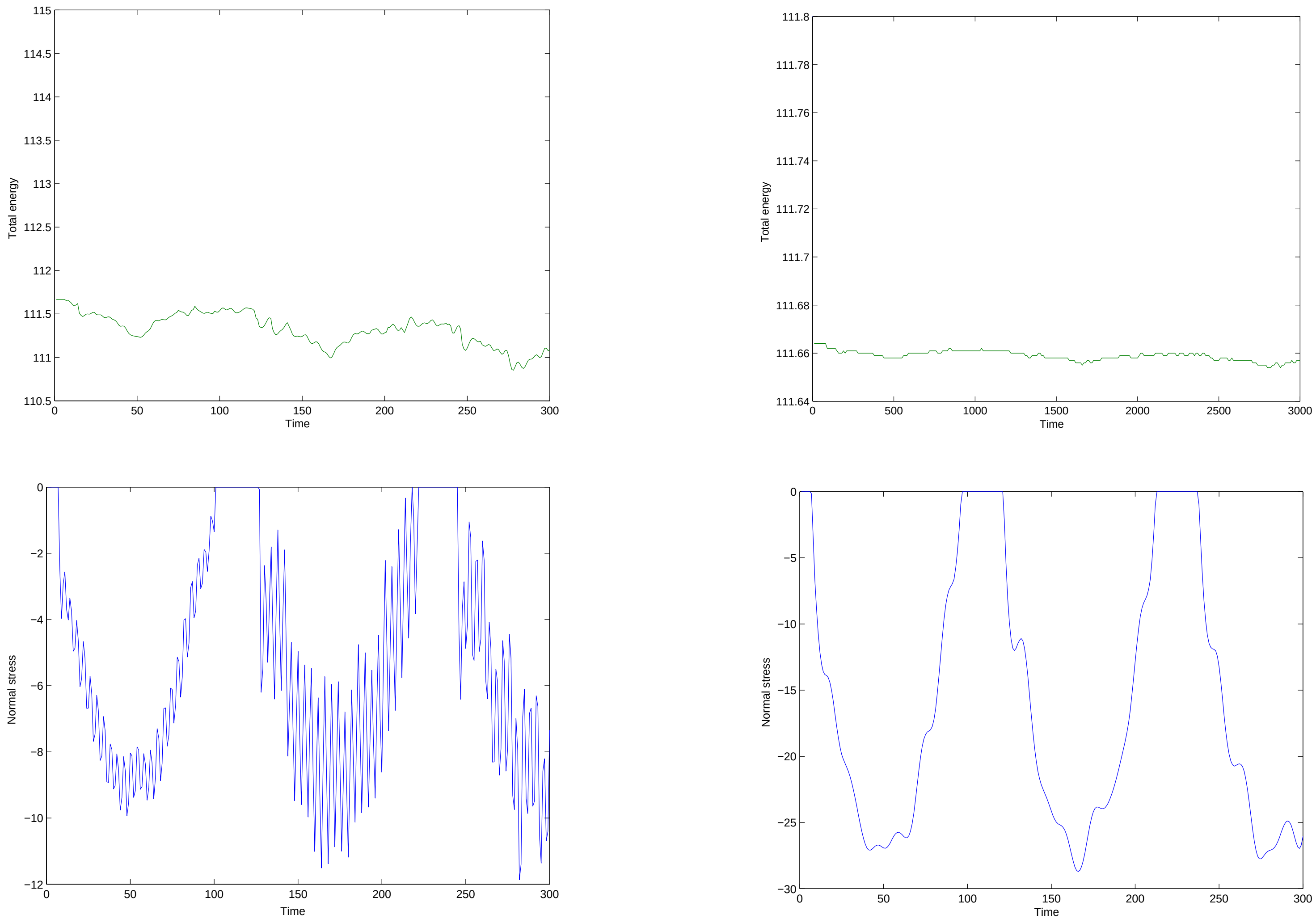
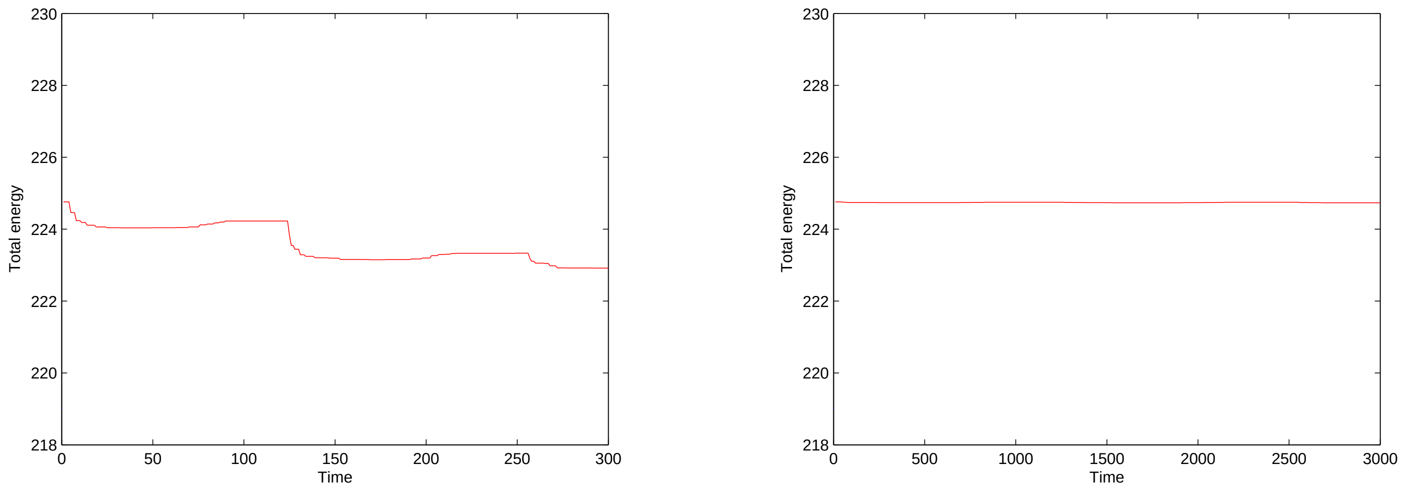


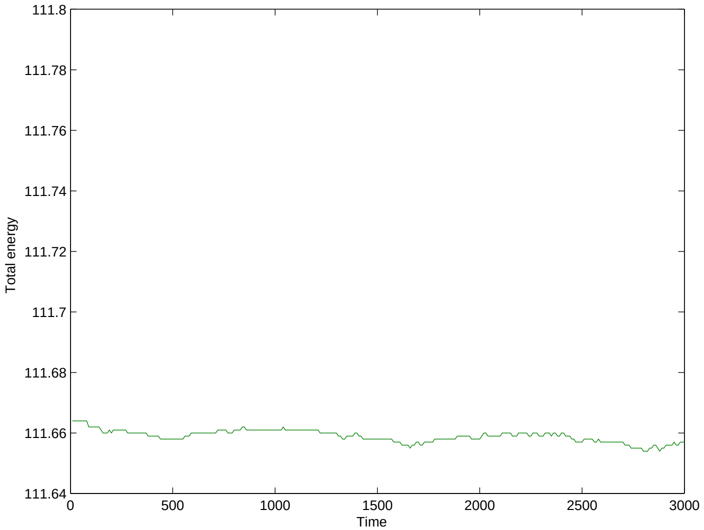
FIG. 6: *energy and normal stress evolution for the Newmark scheme.*

Influence of the time parameter on energy evolution

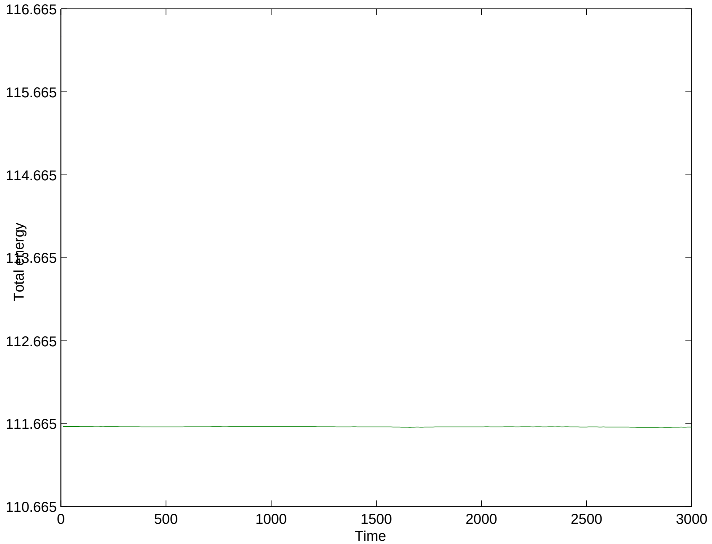


$\Delta t = 0.001$ $\Delta t = 0.0001$

FIG. 7: *Influence of the time parameter on energy evolution for the Crank Nicholson scheme.*



$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$

FIG. 8: *Influence of time parameter on the energy evolution for the Newmark scheme.*

Numerical results

- The redistribution mass method stabilize the Crank Nicholson scheme and Newmark scheme.
- There is some small fluctuations in the energy evolution.
- Energy tends to be conserved when time parameter goes to zero.
- The normal stress in A is more regular for problems expressed with the redistributed mass matrix.

Conclusion

- The space semi-discretized elastodynamic contact problem with the redistributed mass matrix is well posed and is energy conserving.
- The redistribution mass method stabilize classical time integration schemes.
- Energy is conserved when time parameter goes to zero.
- Adding friction condition is not a difficulty from the energy evolution point of vue.