

# ***An energy conserving approximation for elastodynamic contact problems***

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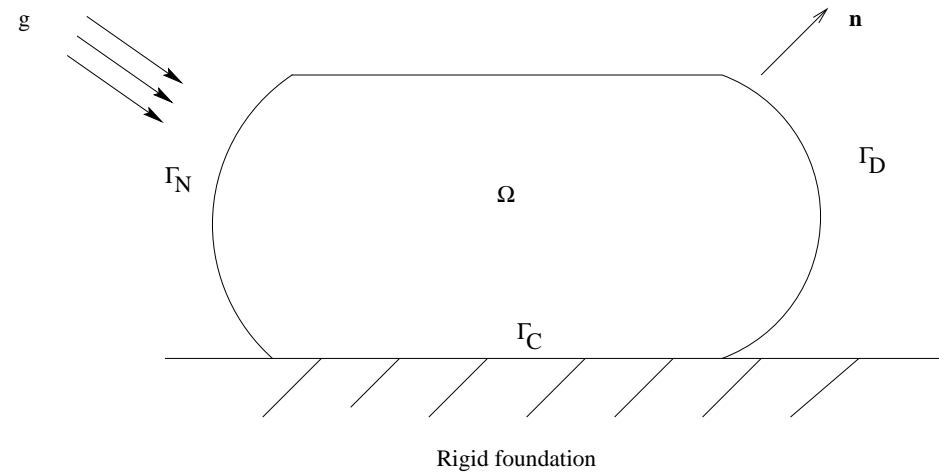
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# Plan

- ▣ Contact problem in elastodynamics
- ▣ Nodal discretization
- ▣ Classical time integration schemes
- ▣ Redistribution Mass Method
- ▣ Numerical results
- ▣ Conclusion



The elastodynamic contact problem reads as : find  $u : [0, T] \longrightarrow \Omega$  such that

$$\rho \ddot{u} - \operatorname{div} \sigma(u) = f, \quad \text{in } ]0, T] \times \Omega,$$

$$\sigma(u) = \mathcal{A} \varepsilon(u), \quad \text{in } ]0, T] \times \Omega,$$

$$\sigma(u) n = g, \quad \text{on } ]0, T] \times \Gamma_N,$$

$$u = 0, \quad \text{on } ]0, T] \times \Gamma_D,$$

$$u(0) = u^0, \dot{u}(0) = v^0, \quad \text{in } \Omega.$$

Assuming that  $\Gamma_C$  has a  $\mathcal{C}^1$  regularity, we have :

$$u_N = u \cdot n, \quad u_T = u - u_N n,$$

$$\sigma_N(u) = (\sigma(u)n) \cdot n, \quad \sigma_T(u) = \sigma(u)n - \sigma_N n.$$

Then, the unilateral contact frictionless condition is expressed as follows :

$$u_N \leq 0, \sigma_N \leq 0, u_N \sigma_N = 0 \text{ and } \sigma_T = 0, \text{ on } ]0, T] \times \Gamma_C.$$

- ➡ Many works have been devoted to the numerical solution of contact problem in elastodynamics.
- ➡ Schatzman-Paoli (2001), Laursen-Chawla (1997, 2002)...
- ➡ For purely contact elastodynamic problems (hyperbolic problems), existence result has been proved in a scalar two dimensional case by Lebeau-Schatzman (1984), Kim (1989) and in the vector case with a modified contact law by Renard-Paumier (2003).

We consider a Lagrange FEM. Let  $a_1, \dots, a_d$  the finite element nodes,  $\varphi_1, \dots, \varphi_d$  be the basis functions and  $I_C = \{i : a_i \in \Gamma_C\}$ . Let  $U$  be the vector of degrees of freedom of the FEM displacement  $u_h(x)$  :

$$u_h(x) = \sum_{1 \leq i \leq d} u_i \varphi_i \quad \text{and} \quad U = (u_i) \in \mathbb{R}^d.$$

Then, the space semi-discretized elastodynamic contact problem reads as follows.

Find  $U : [0, T] \longrightarrow \mathbb{R}^d$  such that

$$\begin{cases} M_0 \ddot{U} + KU = L + \sum_{i \in I_C} \lambda_N^i N_i, \\ \lambda_N^i \leq 0, U \cdot N_i \leq 0, \lambda_N^i (U \cdot N_i) = 0, \forall i \in I_C, \\ U(0) = U^0, \dot{U}(0) = V^0, \end{cases} \quad (1)$$

where

$$K = (K_{ij}) \text{ such that } K_{ij} = \int_{\Omega} \mathcal{A} \varepsilon(\varphi_i) : \varepsilon(\varphi_j) dx,$$

$$L = (L_i) \text{ such that } L_i = \int_{\Omega} f \cdot \varphi_i dx + \int_{\Gamma_C} g \cdot \varphi_i dx,$$

$$M_0 = ((M_0)_{ij}) \text{ such that } (M_0)_{ij} = \int_{\Omega} \rho \varphi_i \cdot \varphi_j dx.$$

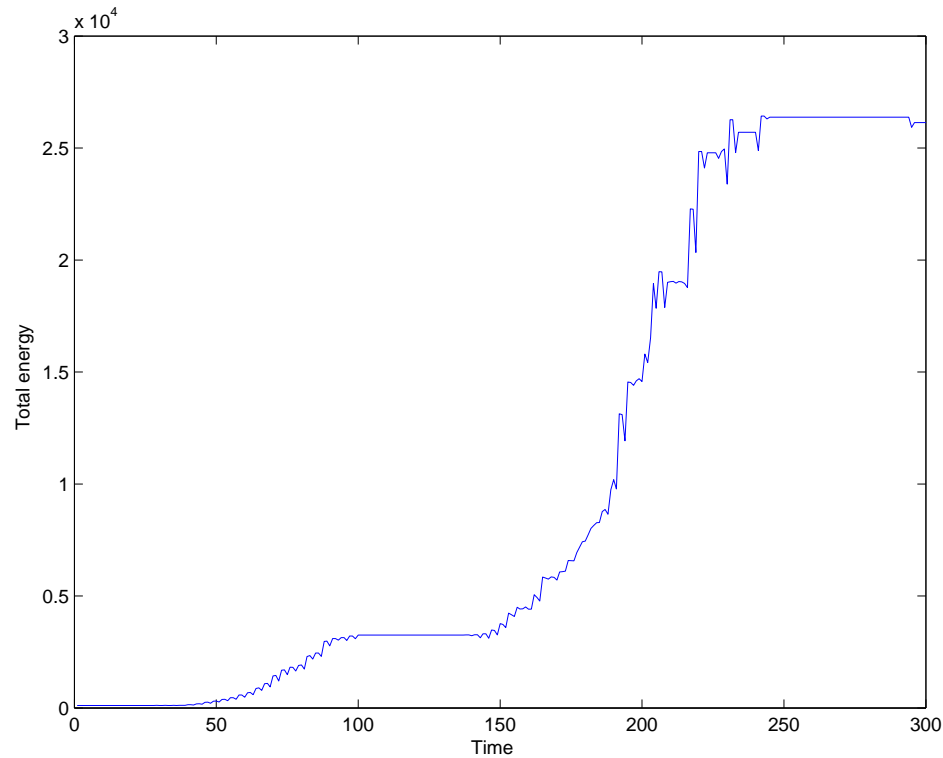
The normal displacement at the node  $a_i$  on the contact surface is equal to :  $u_N^h(a_i) = N_i^T U$ .

The multipliers  $\lambda_N^i$  define the nodal contact forces vector :  $\Lambda_N = (\lambda_N^i)$ .

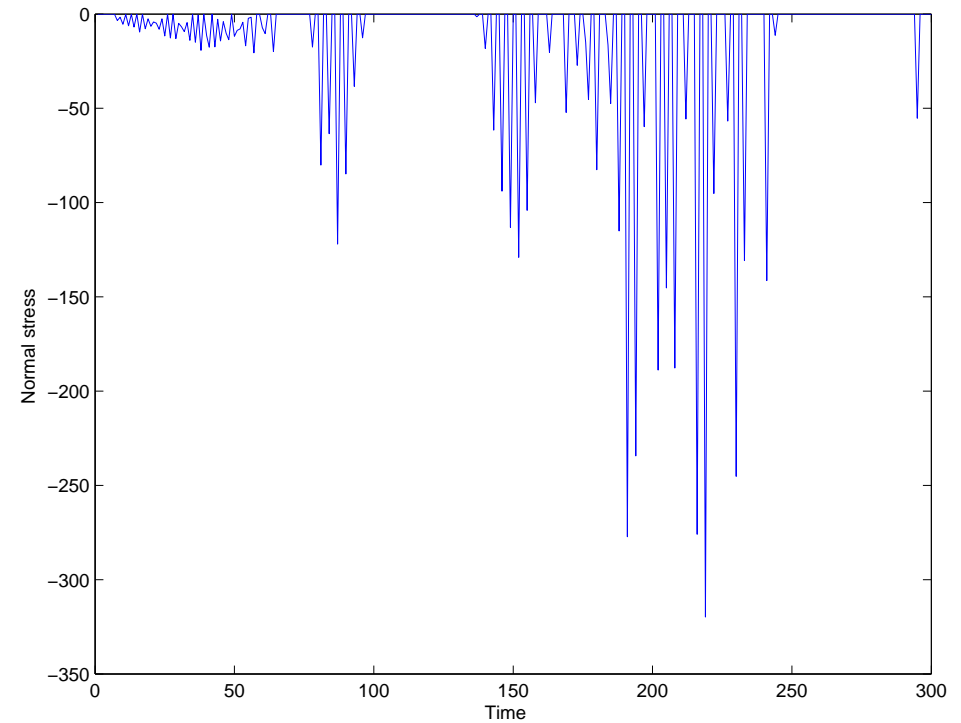
## Remark :

- ▣▶ Problem (1) is ill-posed (Moreau 1983, 1986)
- ▣▶ Uniqueness can be recovered, for rigid bodies, by introducing an impact law with a restitution coefficient (Moreau 1999, Paoli-Schatzman 1993)
- ▣▶ This seems not to be completely satisfying for deformable bodies because whatever the restitution coefficient value is, the system tends to a global restitution of energy when the mesh parameter goes to zero.

The  $\theta$ -method  $\theta = 0.5$  :



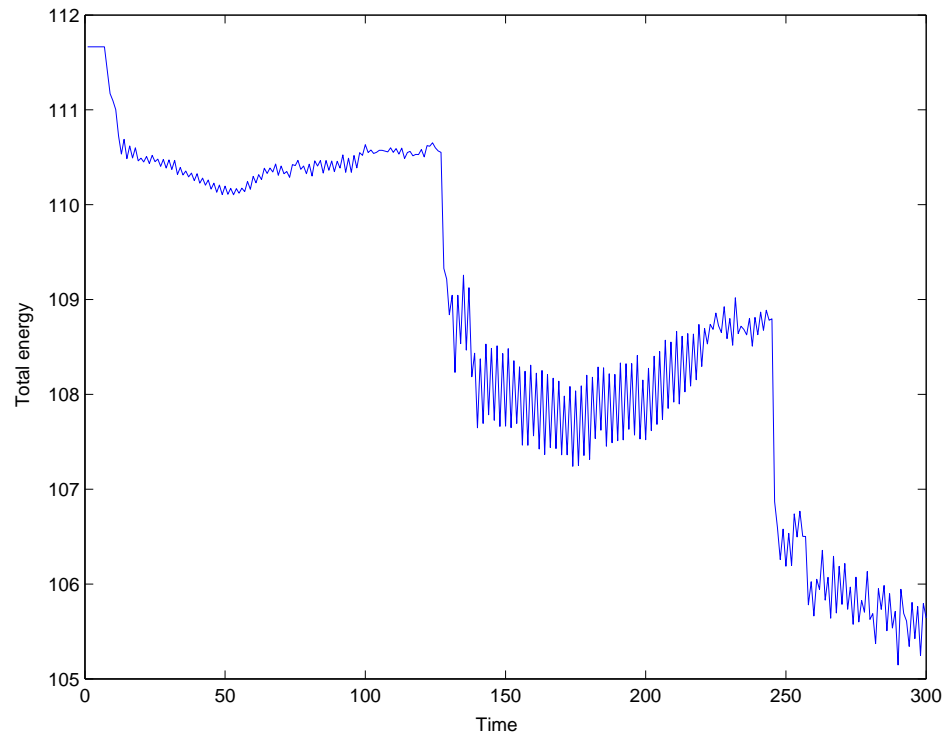
Energy



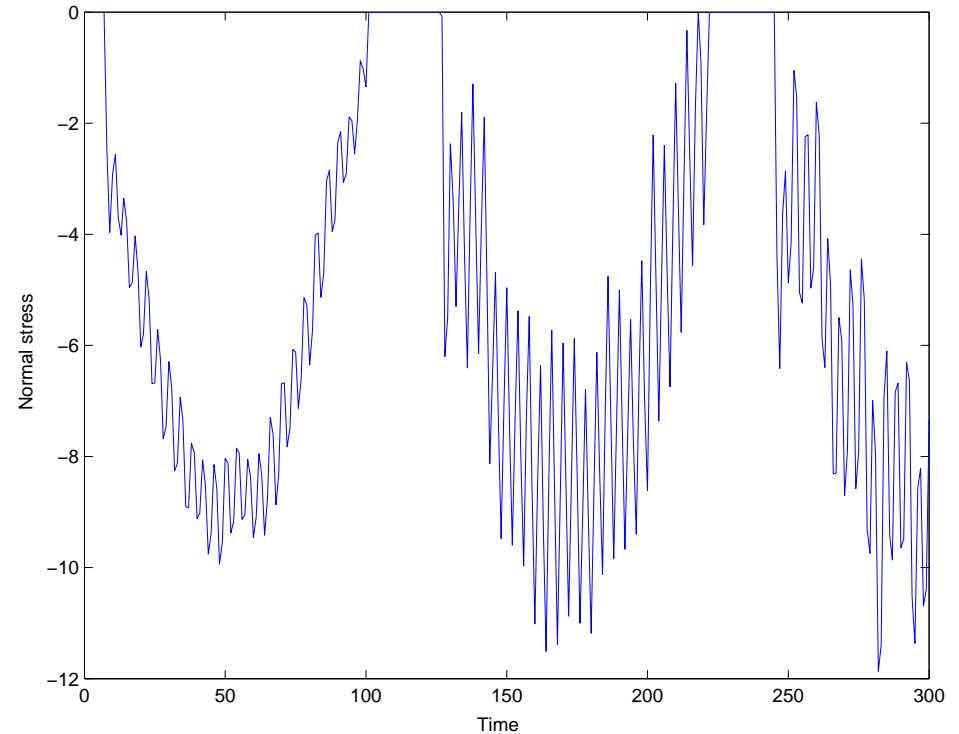
Normal stress

FIG. 1: *energy and normal stress evolution for the  $\theta$ -method.*

The Newmark scheme  $\beta = \gamma = 0.5$  :



Energy

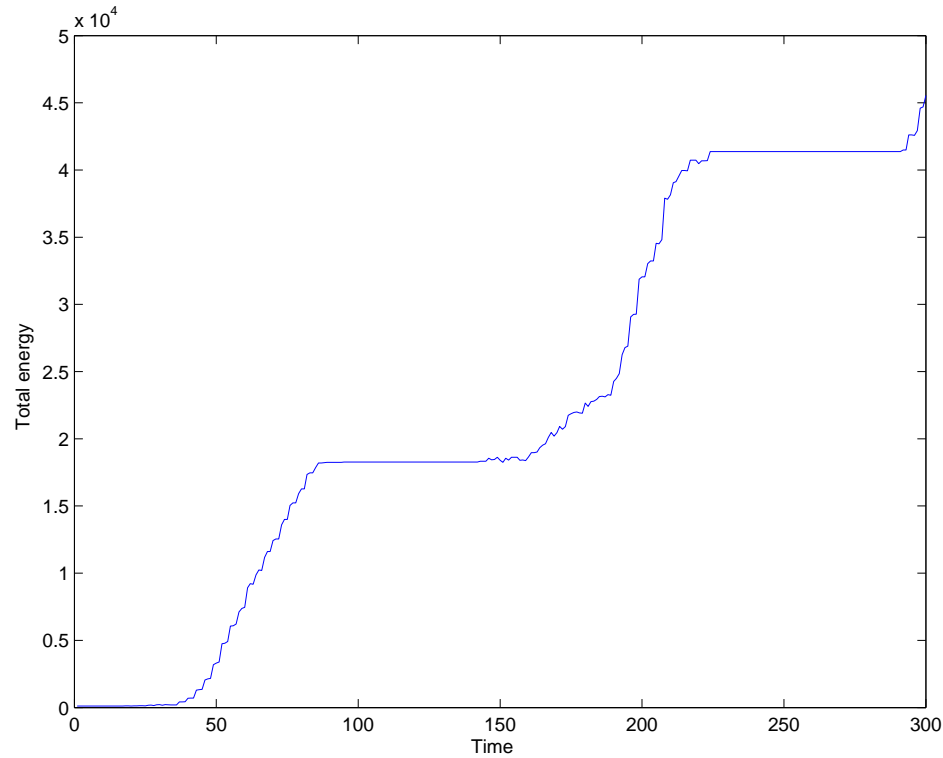


Normal stress

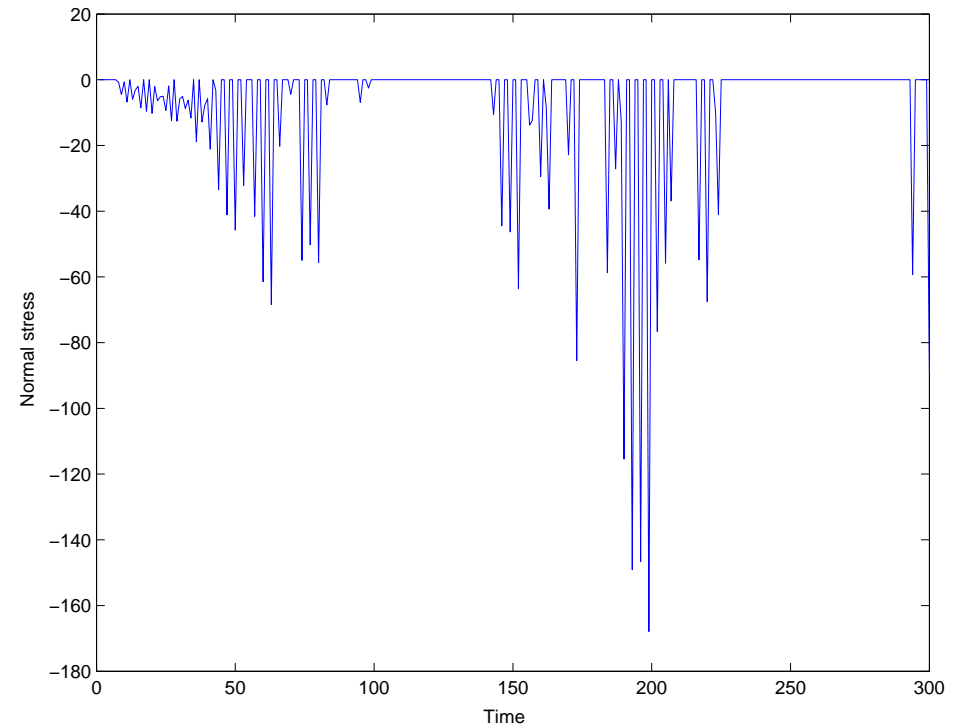
FIG. 2: *energy and normal stress evolution for the Newmark scheme.*



## The midpoint scheme :

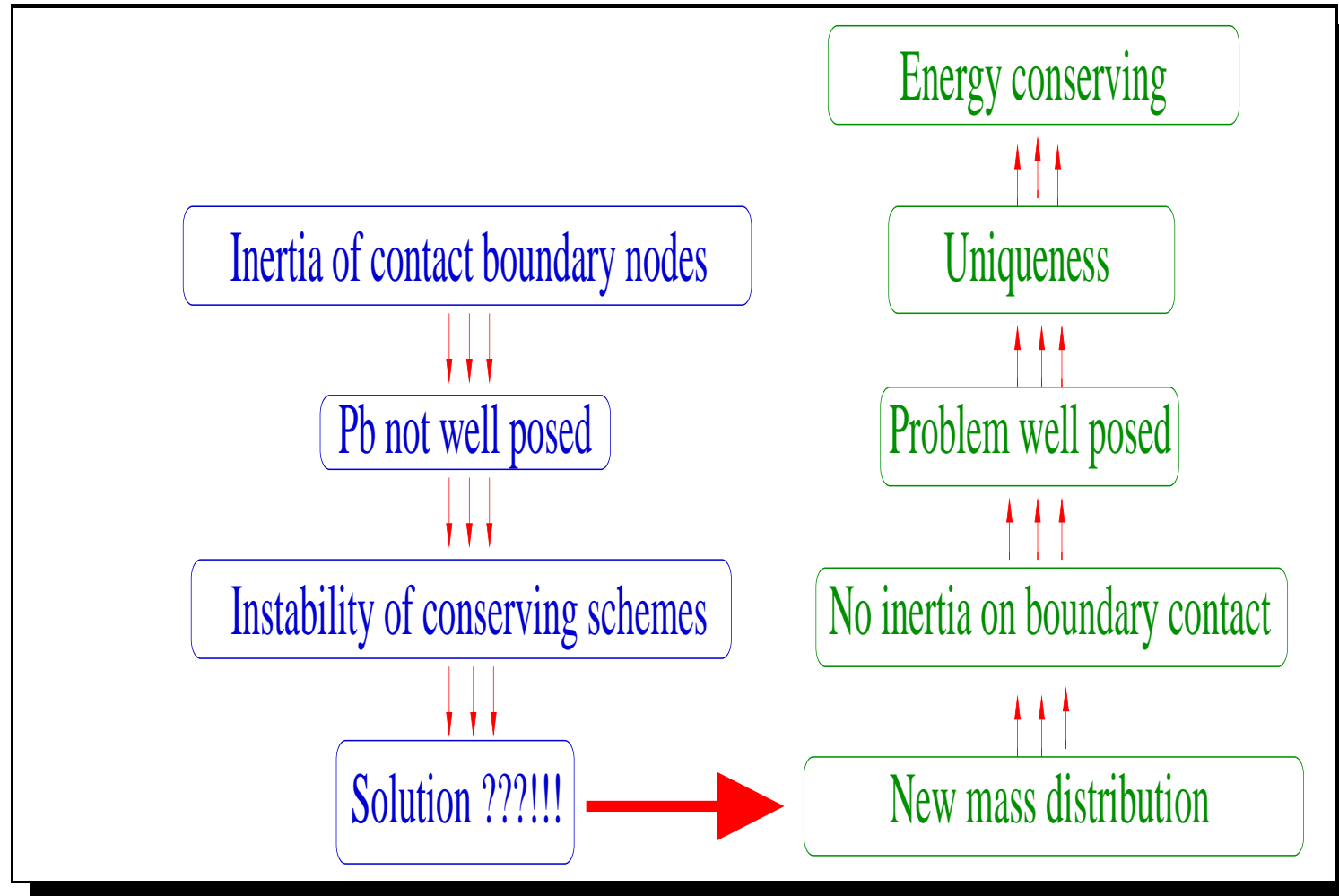


Energy



Normal stress

FIG. 3: *energy and normal stress evolution for the midpoint scheme.*



We denote  $M$  the modified mass matrix such that :

$$\left\{ \begin{array}{l} \inf \\ N_i^T M N_j = 0, \forall i, j \in I_C, \\ \text{the same total mass,} \\ \text{the same gravity center,} \\ \text{the same momentum inertia.} \end{array} \right. \frac{1}{2} \|M - M_0\|^2 \quad (2)$$

In addition,  $M$  must be symmetric and positive semi-definite like  $M_0$ .

## Theorem 1

Let us suppose that the load vector  $L(t)$  is a Lipschitz continuous function on  $[0, T]$ . Then, there exists one and only one Lipschitz continuous function  $(U, \Lambda_N)$  satisfying the finite element elastodynamic system with contact (1) at each time  $t \in [0, T]$ .

**Proof.** If we numerate the degrees of freedom such that the last ones are the nodes on the contact boundary, we can split each matrix and vector in an interior part and a contact boundary one as follows :

$$M = \begin{pmatrix} \bar{M} & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \bar{K} & C^T \\ C & \tilde{K} \end{pmatrix}, \quad U = \begin{pmatrix} \bar{U} \\ \tilde{U} \end{pmatrix},$$

$$L = \begin{pmatrix} \bar{L} \\ \tilde{L} \end{pmatrix} \text{ and } N_i = \begin{pmatrix} 0 \\ \tilde{N}_i \end{pmatrix}.$$

Then, the system (1) is written :

$$\left\{ \begin{array}{l} \overline{M} \ddot{\overline{U}} + \overline{K} \overline{U} = \overline{L} - C^T \tilde{U}, \\ C \overline{U} + \tilde{K} \tilde{U} = \tilde{L} + \sum_{i \in I_C} \lambda_N^i \tilde{N}_i, \\ \tilde{N}_i^T \tilde{U} \leq 0, \quad \lambda_N^i \leq 0, \quad \lambda_N^i (\tilde{N}_i^T \tilde{U}) = 0 \quad \forall i \in I_C. \end{array} \right. \quad (3)$$

The following sub-system of (3) :

$$\left\{ \begin{array}{l} \tilde{K} \tilde{U} + C \overline{U} = \tilde{L} + \sum_{i \in I_C} \lambda_N^i \tilde{N}_i, \\ \tilde{N}_i^T \tilde{U} \leq 0, \quad \lambda_N^i \leq 0, \quad \lambda_N^i (\tilde{N}_i^T \tilde{U}) = 0 \quad \forall i \in I_C \end{array} \right. \quad (4)$$

can be expressed as follows :

$$a(\tilde{U}, \tilde{V} - \tilde{U}) \geq l_{\overline{U}} (\tilde{V} - \tilde{U}) \quad \forall \tilde{V} \in Q, \quad (5)$$

where

$$a(\tilde{U}, \tilde{V}) = \tilde{V}^T \tilde{K} \tilde{U}, \quad l_{\bar{U}}(\tilde{V}) = \tilde{V}^T \tilde{L} - \tilde{V}^T C \bar{U} \quad \text{and} \quad Q = \{V : \tilde{N}_i^T \tilde{V} \leq 0, i \in I_C\}.$$

- $\tilde{U}$  is uniquely defined from the variational inequality (5) for a given  $\bar{U}$ .
- $\tilde{U}$  is a Lipschitz continuous function of  $\bar{U}$ .
- The first equation in the system (3) is a second order Lipschitz ODE ( $\bar{U}$ ).
- This equation has a unique solution  $\bar{U}$  with a Lipschitz continuous derivative.
- There also exists one and only one Lipschitz continuous function in time  $\tilde{U}$  solution to the variational inequality (5).
- The second equation in the system (1) allows us to define a unique Lipschitz continuous function  $\lambda_N^i$  ( $i \in I_C$ ) in time.

## Proposition 1

The solution  $(U, \Lambda_N)$  to the space semi-discretized problem (1) satisfies the following condition at each node on  $\Gamma_C$  :

$$\lambda_N^i (N_i^T \dot{U}) = 0 \quad \text{a.e. on } (0, T) \quad (i \in I_C). \quad (6)$$

**Remark** Formulation (6) is the so-called persistence contact condition in elastodynamics.

**Proof.** Using system (1) and the complementary conditions on the contact area, we have :

$$\lambda_N^i = 0 \quad \text{on} \quad \text{Supp}(N_i^T U) = \omega_i \subset [0, T] \quad (i \in \Gamma_C).$$

The continuity of  $\lambda_N^i$  on  $[0, T]$  implies that :  $\lambda_N^i = 0$  on  $\overline{\omega_i}$ .

On the other hand,  $N_i^T \dot{U} = 0$  a.e. on  $\theta_i$ , where  $\theta_i$  is the complementary part in  $[0, T]$  of the interior of  $\omega_i$ . Hence  $\lambda_N^i (N_i^T \dot{U}) = 0$ , a.e. on  $(0, T)$ .

## Theorem 2

Assuming that the load vector  $L$  is constant in time, the finite element elastodynamic system with unilateral contact (1) is energy conserving.

**Proof.** The discrete energy of system (1) is given by :

$$E(t) = J(U, \dot{U}) = \frac{1}{2} \dot{U}^T M \dot{U} + \frac{1}{2} U^T K U - U^T L.$$

The first equation in (1) implies :

$$\frac{1}{2} \dot{U}^T M \ddot{U} + \frac{1}{2} \dot{U}^T K U = \dot{U}^T L + \sum_{i \in I_C} \lambda_N^i \dot{U}^T N_i.$$

Integrating from 0 to  $t$ , it follows :

$$\frac{1}{2} \dot{U}^T M \dot{U} + \frac{1}{2} U^T K U - U^T L = \sum_{i \in I_C} \int_0^t \lambda_N^i \dot{U}^T N_i ds + E(0).$$



In other words, one has :

$$E(t) = \sum_{i \in I_C} \int_0^t \lambda_N^i \dot{U}^T N_i dt + E(0) \quad \forall t \in [0, T].$$

Thanks to Proposition 1, we finally obtain :

$$E(t) = E(0) \quad \forall t \in [0, T].$$

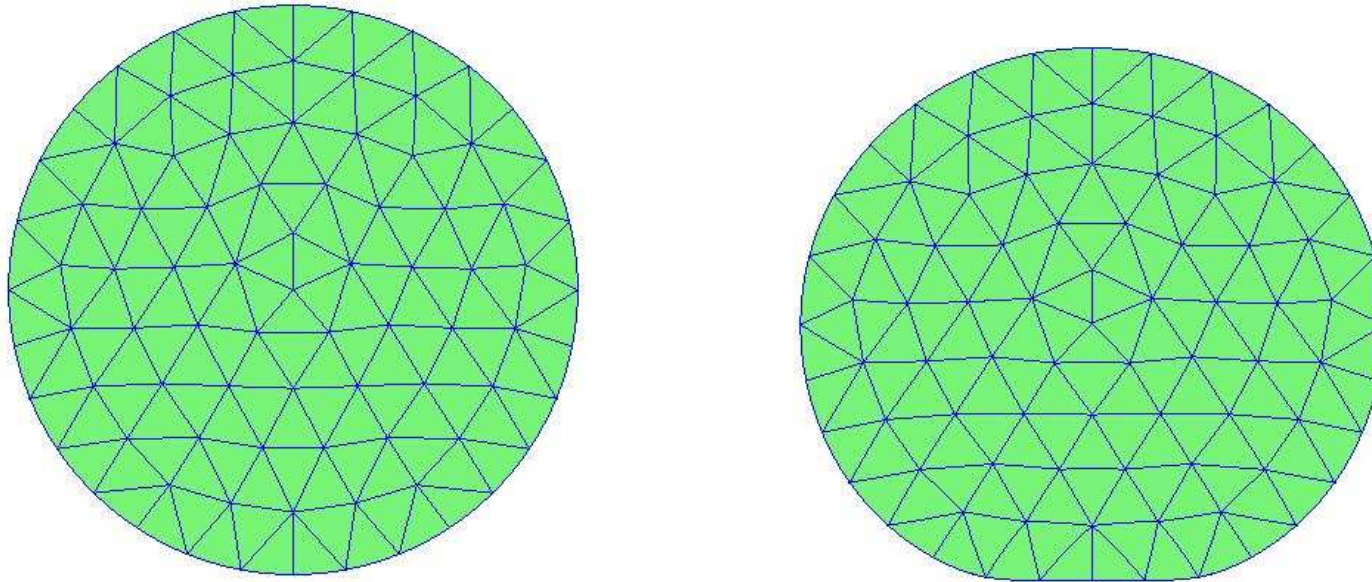


FIG. 4: *a disc before and during the first contact.*

We denote  $A$  the lowest point of the disc (the first point being in contact).

Disc and the resolution method properties	Values
$\rho$ , diameter	$6 \cdot 10^{-6} \text{ kg/cm}^3$ , $20 \text{ cm}$
Lamé coefficients	$\lambda = 10 \text{ GP}$ , $\mu = 5 \text{ GP}$
$u^0$ , $v^0$	$1 \text{ cm}$ , $-100 \text{ cm/s}$
Time parameter	$10^{-3} \text{ s}$
Simulation time	$0.3 \text{ s}$
Mesh parameter	$\simeq 2 \text{ cm}$

TAB. 1: *characteristics of the elastic disc and of the resolution method*

With the standard mass matrix

With the redistributed mass matrix

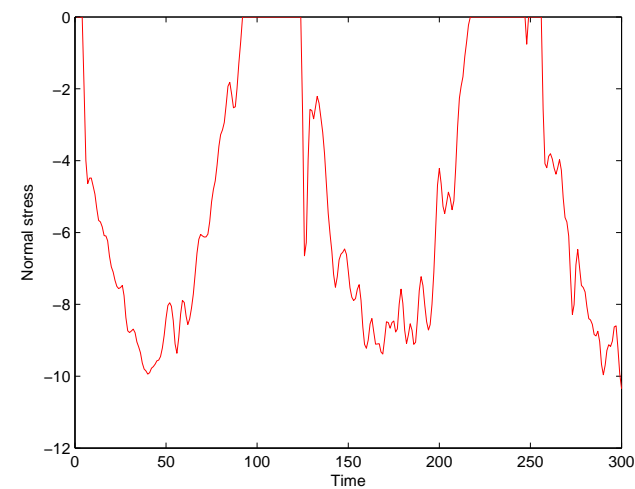
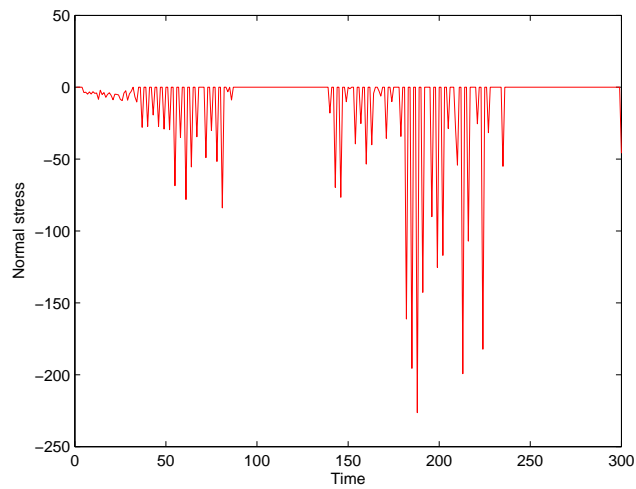
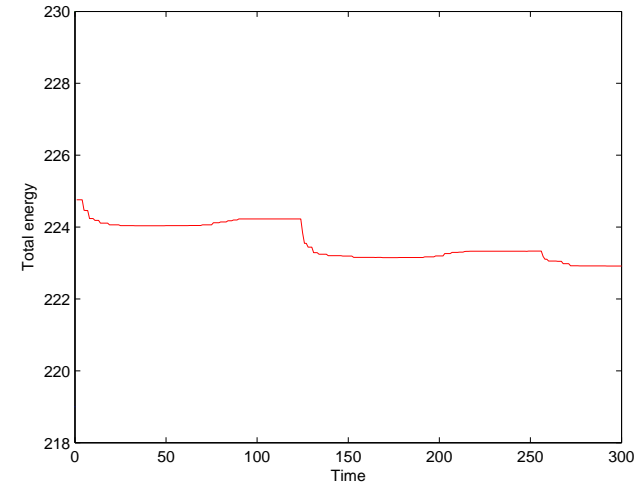
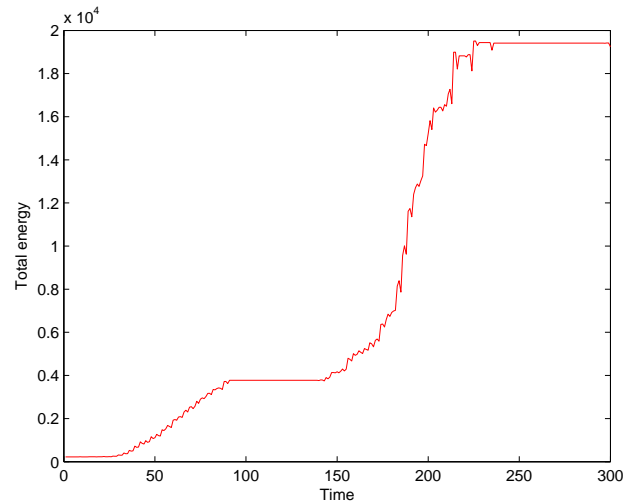


FIG. 5: *energy and normal stress evolution for the Crank Nicholson scheme.*

With the standard mass matrix

With the redistributed mass matrix

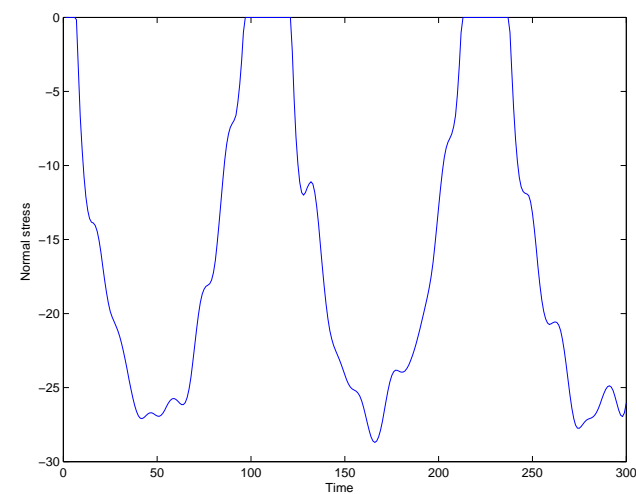
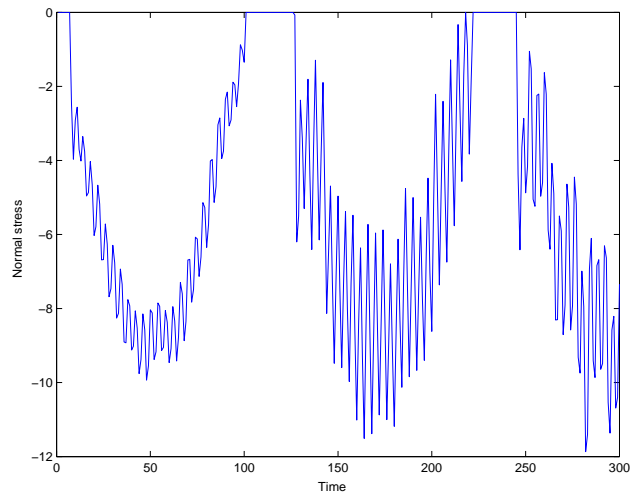
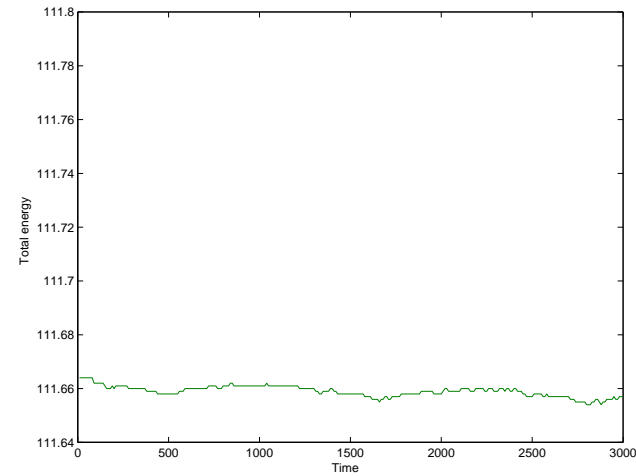
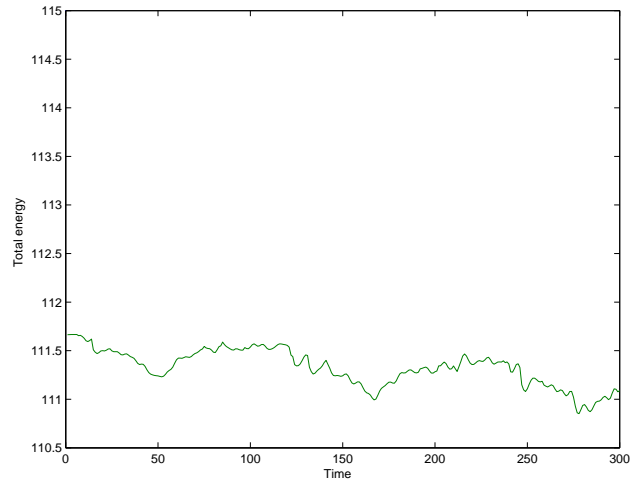
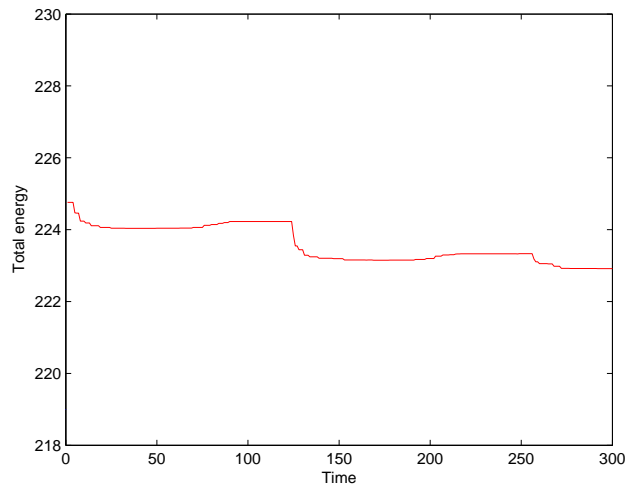
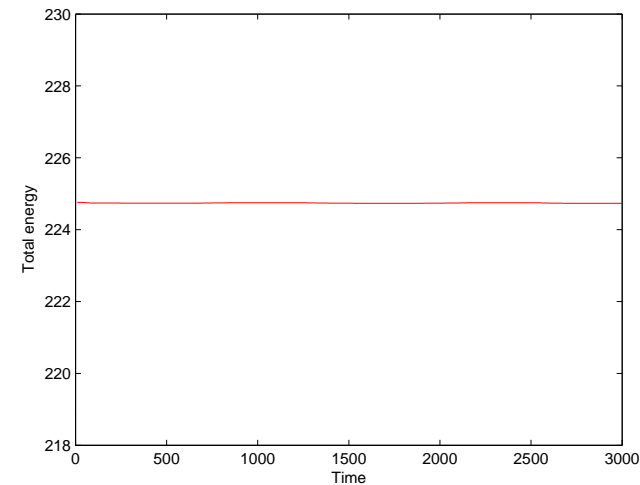


FIG. 6: *energy and normal stress evolution for the Newmark scheme.*

## Influence of the time parameter on energy evolution

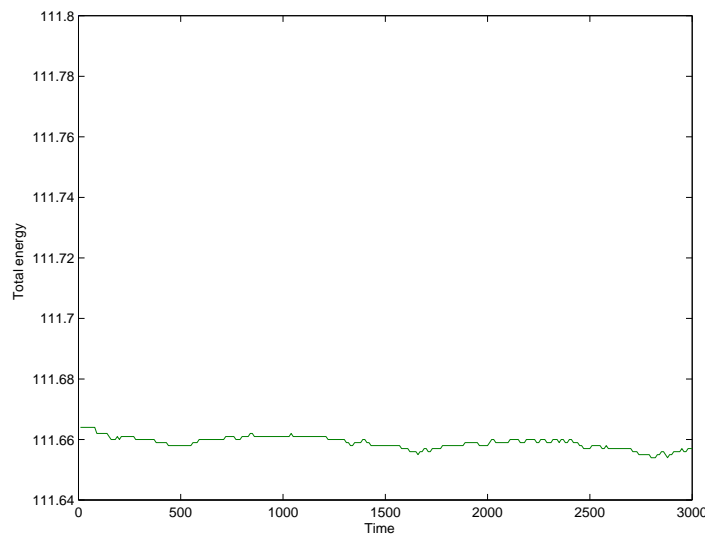


$$\Delta t = 0.001$$

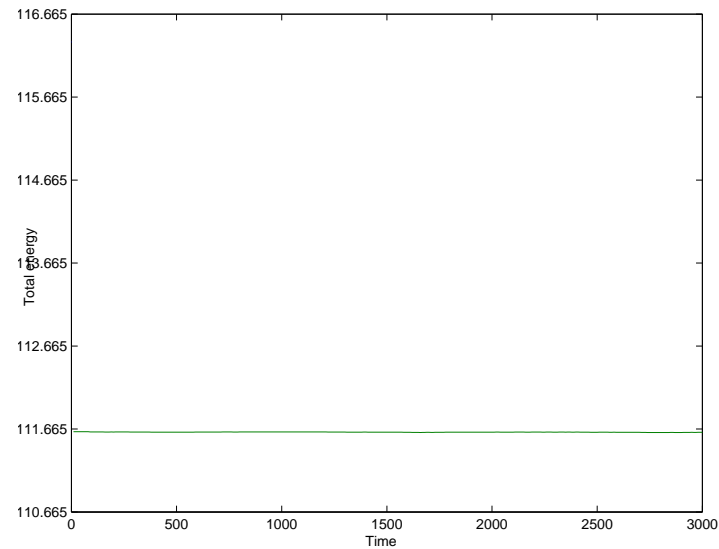


$$\Delta t = 0.0001$$

FIG. 7: *Influence of the time parameter on energy evolution for the Crank Nicholson scheme.*



$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$

FIG. 8: *Influence of time parameter on the energy evolution for the Newmark scheme.*

- The redistribution mass method stabilize the Crank Nicholson scheme and Newmark scheme.
- There is some small fluctuations in the energy evolution.
- Energy tends to be conserved when time parameter goes to zero.
- The normal stress in  $A$  is more regular for problems expressed with the redistributed mass matrix.



## Conclusion

- The space semi-discretized elastodynamic contact problem with the redistributed mass matrix is well posed and is energy conserving.
- The redistribution mass method stabilize classical time integration schemes.
- Energy is conserved when time parameter goes to zero.
- Adding friction condition is not a difficulty from the energy evolution point of view.