Patches of finite elements for singular solutions

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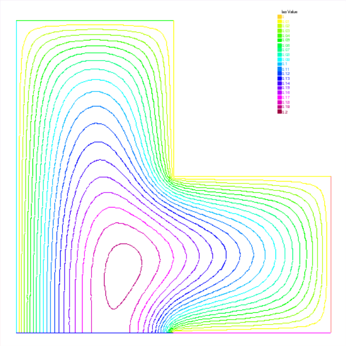
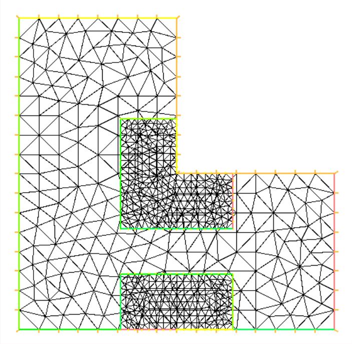
Dirichlet-Neumann BC and domain with entrant corner

L-shaped domain Ω.
Problem of finding $u \in H^1(\Omega)$ s.t. $-\Delta u = 1$ in Ω,
$u = 0$ on $\partial\Omega$, except on Γ where $\frac{\partial u}{\partial n} = 0$.



Γ



Problem setting and approximation

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with boundary $\partial\Omega$ and consider a Galerkin approximation of a problem of the form

$$\begin{cases} \mathcal{L}(u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\mathcal{L}(\cdot)$ is a symmetric strongly elliptic operator. For instance

$$\mathcal{L}(u)(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right).$$

Here $a_{ij} \in L^\infty(\Omega)$, $a_{ij}(x) = a_{ji}(x)$, $1 \leq i,j \leq d$, and

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \forall x \in \Omega.$$

A Galerkin approximation consists:

1. Build a finite dimensional subspace

$$V_{Hh} \subset H_0^1(\Omega).$$

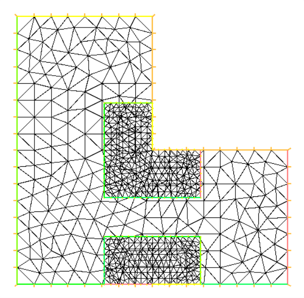
2. Solve the problem:

   Find $u_{Hh} \in V_{Hh}$ satisfying

$$a(u_{Hh}, \varphi) = \langle f | \varphi \rangle, \quad \forall \varphi \in V_{Hh},$$

where

$$a(\psi, \varphi) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial \psi}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx.$$



Two-scale situation

The solution is very sharp in a small subdomain Λ of Ω, but smooth in $\Omega \setminus \Lambda$.

Set $V_{Hh} = V_H + V_h$ where

$$V_H = \{\psi \in H^1_0(\Omega) : \psi|_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_H\}, \text{ and}$$
$$V_h = \{\psi \in H^1_0(\Omega) : \psi|_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_h \text{ and } \psi = 0 \text{ in } \Omega \setminus \Lambda\}.$$

Remark

A priori $V_H \cap V_h \neq \{0\}$ and it is not obvious to exhibit a FE-type basis of V_{Hh}.

Algorithm

- Initialization: $u^0 \in V_H$ s.t. $a(u^0, \varphi) = \langle f | \varphi \rangle, \quad \forall \varphi \in V_H$, and choose $\omega \in (0;2)$.
- For $n = 1, 2, 3, \dots$ find
  1. $w_h \in V_h$ such that
     $$a(w_h, \varphi) = \langle f | \varphi \rangle - a(u^{n-1}, \varphi), \quad \forall \varphi \in V_h \ ;$$
     $u^{n-\frac{1}{2}} = u^{n-1} + \omega w_h$;
  2. $w_H \in V_H$ such that
     $$a(w_H, \varphi) = \langle f | \varphi \rangle - a(u^{n-\frac{1}{2}}, \varphi), \quad \forall \varphi \in V_H \ ;$$
     $u^n = u^{n-\frac{1}{2}} + \omega w_H$.

Implementation of the method

We can rewrite the algorithm:

1. $v_h \in V_h$ s.t. $a(v_h, \varphi) = \langle f|\varphi\rangle - a(u_H^{n-1}, \varphi), \quad \forall \varphi \in V_h$;
   $u_h^n = (1-\omega)u_h^{n-1} + \omega v_h$;
2. $v_H \in V_H$ s.t. $a(v_H, \varphi) = \langle f|\varphi\rangle - a(u_h^n, \varphi), \quad \forall \varphi \in V_H$;
   $u_H^n = (1-\omega)u_H^{n-1} + \omega v_H$.

Hence $u^n = u_H^n + u_h^n$ with $u_H^n \in V_H$, $u_h^n \in V_h$.
(Direct comparison with Chimera method when $\omega = 1$.)
When implementing the algorithm, the coarse and the fine parts of u^n are stored separately.
We use an interpolation technique in order to integrate terms like $a(\varphi_H, \varphi_h)$, $\varphi_H \in V_H$, $\varphi_h \in V_h$.
A fine structured grid is used to define the relationship between $\mathcal{T}_H$ and $\mathcal{T}_h$.

Bibliographical remarks

- Domain decomposition
- Chimera method
  Steger et al. (1987) and after Brezzi et al. (2003), ...
- Multiplicative Schwarz method
  - Fast Adaptive Composite (FAC) grid method
    McCormick (1984) and after McKay, Thomas, Philip, Lee, Quinlan, ...
  - Hierarchical method
    Yserentant (1986), Bank, Dupont, Yserentant (1988), ...
  - Successive Subspace Correction method
    Xu (1992), Zikatanov (2000), ...
  - Method without any conformity between the meshes and complete overlapping: *R. Glowinski et al. (2003).*

The iteration operator B

- If $P_h : V_{Hh} \to V_h$ and $P_H : V_{Hh} \to V_H$ are orthogonal projectors with respect to the scalar product $a(\cdot,\cdot)$, then we have

$$u_{Hh} - u^n = B(u_{Hh} - u^{n-1}),$$

with

$$B = (I - \omega P_H)(I - \omega P_h).$$

- The convergence speed is given by $\rho(B)$.
- The factor of reduction of the error in the norm $a(\cdot,\cdot)^{1/2}$ is bounded by $||B||$.

The parameter $\tilde{\gamma}$

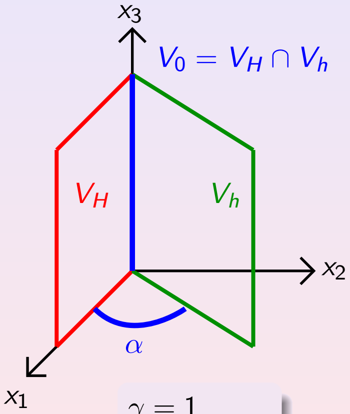
We introduce the number $\gamma \in [0;1]$ s.t.

$$\gamma = \sup_{\substack{v_h \in V_h, v_h \neq 0 \\ v_H \in V_H, v_H \neq 0}} \frac{a(v_h, v_H)}{||v_h||||v_H||}.$$

We set $V_0 = V_H \cap V_h$,
$\tilde{V}_h = V_h \cap V_0^{\perp}$, $\tilde{V}_H = V_H \cap V_0^{\perp}$, and

$$\tilde{\gamma} = \sup_{\substack{v_h \in \tilde{V}_h, v_h \neq 0 \\ v_H \in \tilde{V}_H, v_H \neq 0}} \frac{a(v_h, v_H)}{||v_h||||v_H||},$$

if $V_h \neq V_0$ and $V_H \neq V_0$ (otherwise $\tilde{\gamma} = 0$).



$\gamma = 1$
$\tilde{\gamma} = \cos\alpha$

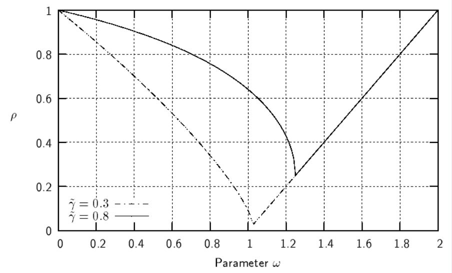
The spectral radius $\rho(B)$

Theorem

For $\omega \in (0;2)$, the spectral radius of B is given by $\rho(B) = \rho(\tilde{\gamma}, \omega)$ where

$$\rho(\tilde{\gamma},\omega) = \begin{cases} \frac{\omega^2\tilde{\gamma}^2}{2} - \omega + 1 + \frac{\omega\tilde{\gamma}}{2}\sqrt{\omega^2\tilde{\gamma}^2 - 4\omega + 4}, & \text{if } \omega \leq \omega_0(\tilde{\gamma}), \\ \omega - 1, & \text{otherwise}, \end{cases}$$

where $\omega_0(\tilde{\gamma}) = \frac{2-2\sqrt{1-\tilde{\gamma}^2}}{\tilde{\gamma}^2}$ for $\tilde{\gamma} \in (0;1]$, and $\omega_0(\tilde{\gamma}) = 1$ for $\tilde{\gamma} = 0$.



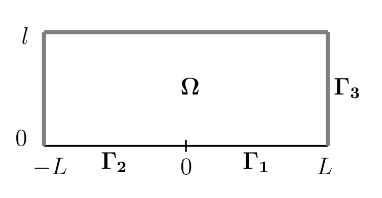
Remarks

- $\tilde{\gamma}$ is necessarily strictly smaller than 1.
- We have $\rho(B) < 1$ and if $\omega = 1$,we have $\rho(B) = \tilde{\gamma}^2$.

For given $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

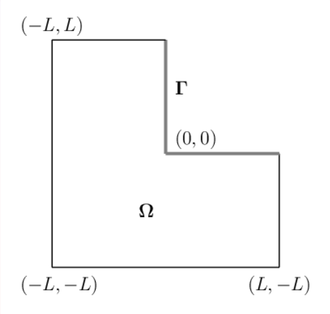
Problem 1

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \\ u = 0 & \text{on } \Gamma_3. \end{cases}$$



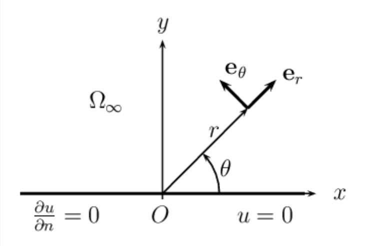
Problem 2

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



Problem 1: Regularity result

In $\Omega_\infty = (-\infty; +\infty) \times (0; +\infty)$,
find $v \in H^1_{\text{loc}}(\overline{\Omega}_\infty)$ such that
$\Delta v = 0$ in Ω_∞,



$$\frac{\partial v}{\partial n} = 0 \quad \text{on } (-\infty; 0) \times \{0\},$$
$$v = 0 \quad \text{on } (0; +\infty) \times \{0\}.$$

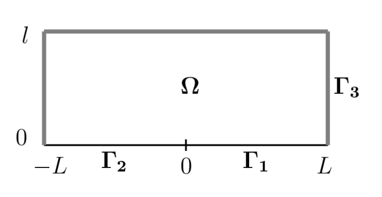
Grisvard $\Rightarrow v \in W^{2,p}_{\text{loc}}(\overline{\Omega}_\infty)$ with $p \in [1; \frac{4}{3})$.

Denote $\varphi(r, \theta) = c_0 \sqrt{r} \sin(\theta/2)$.

Model problem and a priori error estimate

$\Omega = (-L; L) \times (0; l) \subset \mathbb{R}^2$
For given $f \in L^2(\Omega)$,
find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \\ u = 0 & \text{on } \Gamma_3. \end{cases}$$



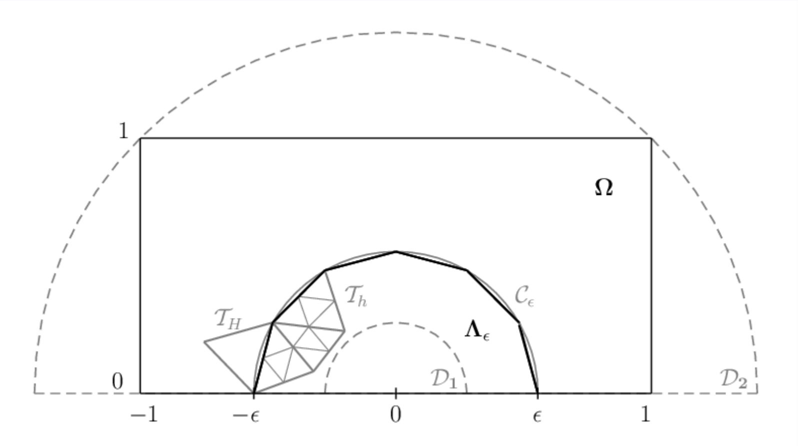
$u = w + \varphi$ where $w \in H^2(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$, $p \in [1; \frac{4}{3})$.

For $p \in (1; 4/3)$, $\exists H_0$ such that

$$||u - u_H||_{H^1(\Omega)} \leq CH^{2-2/p} |u|^{p/2}_{W^{2,p}(\Omega)}, \quad \forall H \leq H_0,$$

C is independent of H and u but depends on p.

Improving the convergence order with patches



Proposition

Let $p \in (1;4/3)$ and consider the setting illustrated below. Then there exist C and h_0 such that the approximation u_{Hh} to u satisfies the *a priori* error estimate

$$||u - u_{Hh}||_{H^1(\Omega)} \leq C \left(\frac{H}{\sqrt{\epsilon}} + h^{2-2/p} + \frac{H^2}{\epsilon h^{1/2}} \right), \quad \forall h \leq h_0, H/\epsilon \to 0,$$

C and h_0 are independent of H, h, ϵ, but depend on p and u.

Skip proof

With p close to $4/3$, $\epsilon \propto H^\beta$, $\beta < 1$,

$$||u - u_{Hh}||_{H^1(\Omega)} \sim \left(H^{1-\beta/2} + h^{1/2} + H^{2-\beta}/h^{1/2} \right),$$

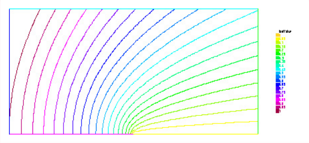
hence choose $h \propto H^{2-\beta}$ and convergence order in H is $\to 1 - \beta/2$.

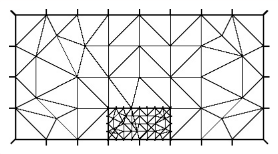
Test problem

We assume that the previous Proposition is true in a more general framework.
Consider unstructured triangulations, a rectangular patch and the problem of finding $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \\ u = \varphi|_{\Gamma_3} & \text{on } \Gamma_3, \end{cases}$$

where $\varphi(r, \theta) = \sqrt{r}\sin(\theta/2)$.





Numerical results

ϵ	h	M	CO1	CO2
0.25	H^2	$N^2/32$	1	1.06
$H^{1/4}$	$H^{7/4}$	$N^{3/2}/\sqrt{2}$	$7/8 = 0.875$	0.85
$\sqrt{H}$	$H^{3/2}$	N	$3/4 = 0.750$	0.72
$H^{3/4}$	$H^{5/4}$	$N^{1/2}\sqrt{2}$	$5/8 = 0.625$	0.61
	no patch		0.5	0.50

$H^1(\Omega)$-norm convergence orders:
CO1 = extremal *a priori* order in H,
CO2 = order obtained by numerical experience.

Some bibliographical references

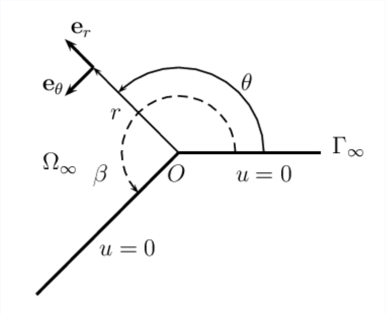
- Babuska (Computing, 1970)
- Brenner (Math. Comp., 1999)
- Dauge (e.g. Springer-book, 1988)
- Grisvard (e.g. Pitman-book, 1985)
- Kim and Cai (SIAM J. Numer. Anal., 2001)
- Raugel (e.g. C. R. Acad. Sci. Paris, 1978)
- ...

Problem 2: Regularity result

Find $v \in H^1_{\mathrm{loc}}(\overline{\Omega}_\infty)$ such that

$\Delta v = 0$ in Ω_∞,

$v = 0$ on Γ_∞.



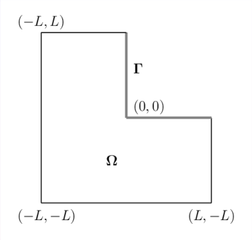
Grisvard $\Rightarrow$ $v \in W^{2,p}_{\mathrm{loc}}(\overline{\Omega}_\infty)$ with $p \in [1; \frac{2}{2-\pi/\beta})$.

For $\beta = 3\pi/2$, denote $\varphi(r,\theta) = c_0 r^{2/3} \sin(2\theta/3)$.

Model problem and a priori error estimate

L-shaped $\Omega \subset (-L; L)^2 \subset \mathbb{R}^2$
For given $f \in L^2(\Omega)$,
find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



$u = w + \varphi$ where $w \in H^2(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$, $p \in [1; \frac{3}{2})$.

For $p \in (1; 3/2)$, $\exists H_0$ such that

$$||u - u_H||_{H^1(\Omega)} \leq CH^{2-2/p} |u|^{p/2}_{W^{2,p}(\Omega)}, \quad \forall H \leq H_0,$$

C is independent of H and u but depends on p.

Improving the convergence order with patches

If $p \in (1; 3/2)$, then there exists C and h_0 such that the approximation u_{Hh} to u satisfies the *a priori* error estimate

$$||u - u_{Hh}||_{H^1(\Omega)} \leq C \left(\frac{H}{\epsilon^{1/3}} + h^{2-2/p} + \frac{H^2}{\epsilon^{5/6} h^{1/2}} \right), \quad \forall h \leq h_0, H/\epsilon \to 0,$$

C and h_0 are independent of H, h, ϵ, but depend on p and u.

With p close to $3/2$, $\epsilon \propto H^{\beta}$,$\beta < 1$,

$$||u - u_{Hh}||_{H^1(\Omega)} \sim \left(H^{1-\beta/3} + h^{2/3} + H^{2-5\beta/6}/h^{1/2} \right),$$

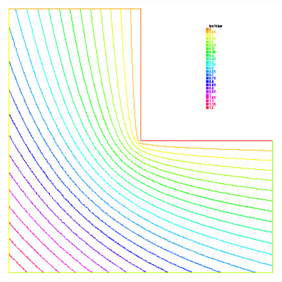
hence choose $h \propto H^{3/2-\beta/2}$ and conv. order in H is $\to 1 - \beta/3$.

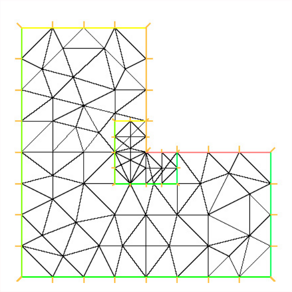
Test problem

Consider unstructured triangulations and the problem of finding $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi|_{\partial\Omega} & \text{on } \partial\Omega, \end{cases}$$

where $\varphi(r, \theta) = r^{2/3} \sin(2\theta/3)$.





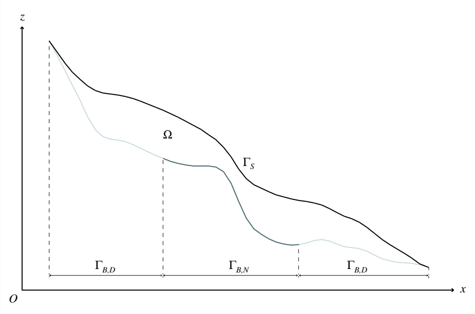
Numerical results

ϵ	h	M	CO1	CO2
0.25	$H^{3/2}$	$N^{3/2}/2^{7/2}$	1	0.93
$H^{1/3}$	$H^{4/3}$	N	$8/9 \approx 0.89$	0.79
$H^{2/3}$	$H^{7/6}$	$N^{1/2}\sqrt{2}$	$7/9 \approx 0.78$	0.74
no patch			$2/3 \approx 0.67$	0.63

$H^1(\Omega)$-norm convergence orders:
CO1 $=$ extremal *a priori* order in H,
CO2 $=$ order obtained by numerical experience.

Two-dimensional glacier

Gries glacier (Swiss Alps)



Glacier model and parameters by Blatter and Funk (ETH Zurich).

Horizontal velocity u – Ice as an incompressible viscous fluid, FOA

Find u defined in Ω such that

$$\begin{cases} -\mathrm{div}(\mu(|\nabla u|)\nabla u) = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_S, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_{B,N}, \\ u = 0 & \text{on } \Gamma_{B,D}, \end{cases}$$

where $f = -\frac{1}{4}\rho g \frac{\partial S}{\partial x}$, and

$$A\left([\mu(s)]^{n-1}s^{n-1} + \sigma_0^{n-1}\right) = \frac{1}{2\mu(s)},$$

with $A = 0.08$ and $\sigma_0 = 0.1$, and $n \geq 1$ an exponent.

Weak formulation

Find $u \in V = \{\varphi \in W^{1,p}(\Omega) : \varphi = 0 \text{ on } \Gamma_{B,D}\}, p = \frac{n+1}{n}$, such that

$$\int_\Omega \mu(|\nabla u|)\nabla u \cdot \nabla \varphi \, \mathrm{d}\Omega = \int_\Omega f \varphi \, \mathrm{d}\Omega, \qquad \forall \varphi \in V.$$

Discretization and linearization

We use linear finite elements and Picard's iterative method, proved to converge in the discrete case (Reist, 2005).

Effective stress field

$$\sigma_{(II)} = \mu(|\nabla u|)|\nabla u| \in L^2(\Omega).$$

Grid constellation ($N = 50$)

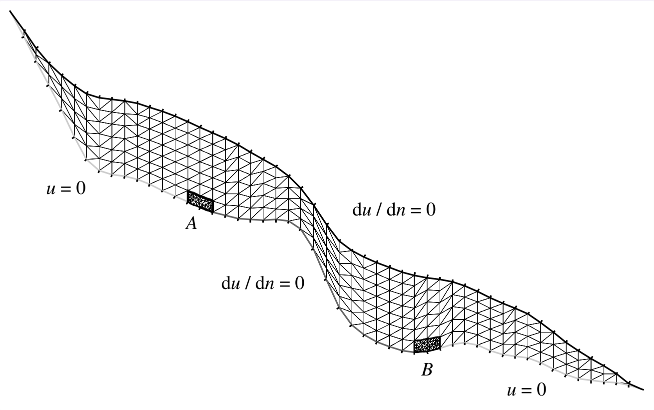
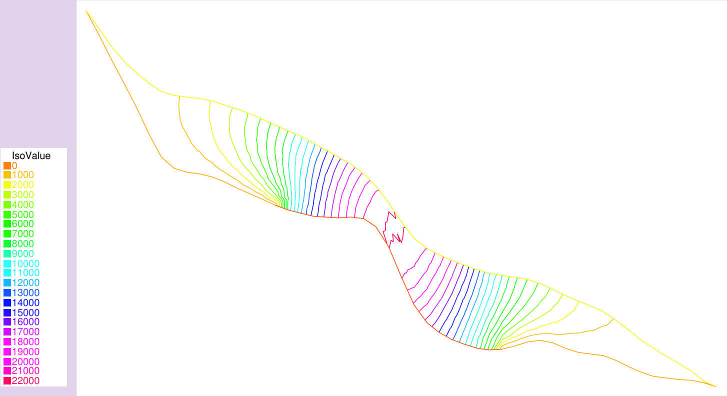
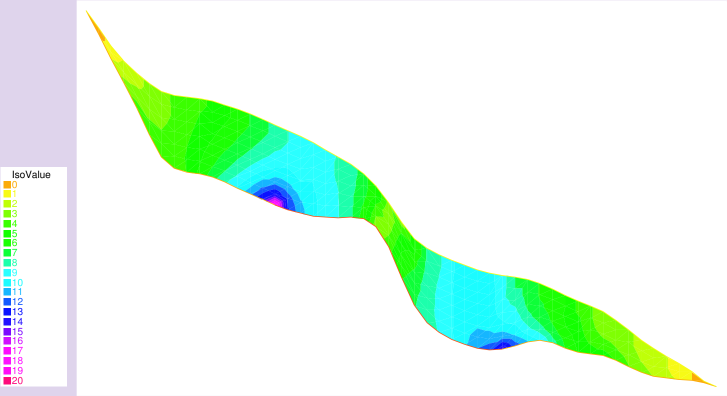


Illustration of two patches ($H/h = 4$) around the points A and B.

Velocity field (nonlinear case, no patch)



Stress field (nonlinear case, no patch)

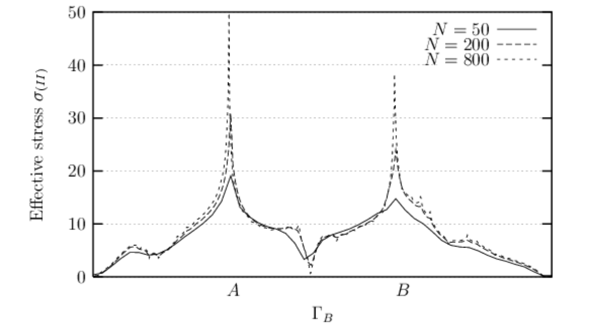


H^1- resp. $W^{1,3/2}$-norm relative error on the velocity field

no patch, N	Linear problem ($n = 1$)	Nonlinear problem ($n = 2$)
50	0.261	0.235
100	0.204	0.154
200	0.172	0.0996
400	0.111	0.0535
800	0.0605	0.0223
1600	0.0463	0.0104
obtained order	$O(H^{0.50})$	$O(H^{0.90})$

Linear problem, D.-N. B.C.: $u \in W^{2,r}(\Omega)$, $r \in [1;4/3)$
$\Rightarrow u \in W^{1,\sigma}(\Omega)$, $\sigma \in [1;4)$, i.e. $u \in H^1(\Omega)$; CO in $H^1 < 0.5$
Non-linear probl.: $u \in W^{1,3/2}$, CO in $H^{0.9}$, u probably in $W^{2,3/2}$!?

Stress field $\sigma_{(II)}(u_H)$ on Γ_B



Heuristic evaluation of the singularity:

$\mu(|\nabla u|) \propto |\nabla u|^{1/n-1}$: $n = 2 \Rightarrow$ (Ansatz) $\mu \propto r^{1/2-\alpha/2}$

$\text{div}(\mu\nabla u)) = 0 \Rightarrow \sigma_{(II)} = \mu(|\nabla u|)|\nabla u| \approx \frac{C}{r^{0.32}}$

Grid constellation ($N = 50$)

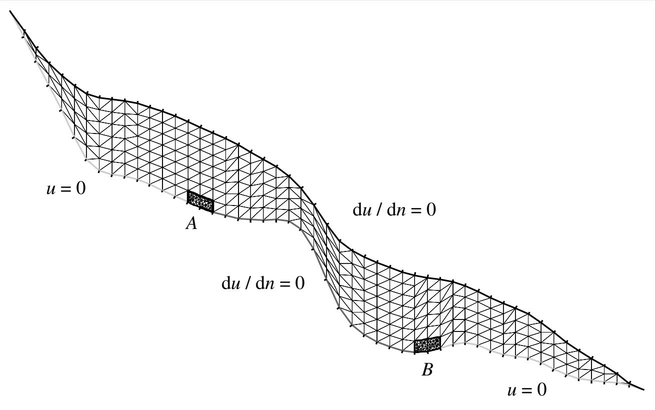
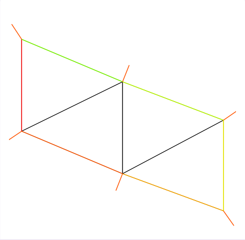
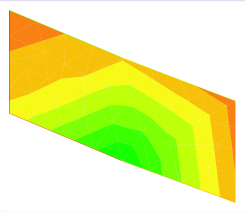


Illustration of two patches ($H/h = 4$) around the points A and B.

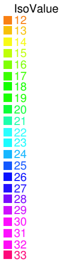
Stress field in the patched region around A ($N = 50$, $H/h = 4$)

Coarse mesh





Relative L^2-error in the patched region divided by 2!



Patch