

# Numerical Method for Elliptic Multiscale Problems

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## OUTLINE

- 1. Introduction**
- 2. Multiscale approach**
- 3. Theory for periodic problems**
- 4. Numerical Experiment**

## Introduction

Multiscale problems are described by partial differential equations with highly oscillatory coefficients.

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{with} \quad L = - \sum_{i,j=1}^d \frac{\partial}{\partial_j} \alpha_{ij} \frac{\partial}{\partial_i}.$$

**Difficulty:** accurate discrete solution of such problems requires a very fine discretisation.

⇒ High storage and computation costs.

**Interest:** the average behaviour of the elliptic oscillatory operator on a coarse scale taking into account the small scale features of the solution.

## Different types of multiscale methods

1. Heterogeneous Multiscales Method [Weinan\Engquist,03]
2. Multiresolution Methods [Brewster\Beylkin, 95].
3. Generalised Finite Element Method: Babuška-Osborn,1d, [83]. Generalised to 2d by Hou-Wu [97].
4. Variational multiscale approach introduced by Hughes and Brezzi, Arbogast for a mixed variant.

**Our approach:** to provide a smoother elliptic operator which behaves like the original operator on a coarse mesh, with no smoothness or periodicity requirement.

## 1D Elliptic Problem

Let  $\Omega = (0, 1)$  and  $V = H_0^1(\Omega)$ . Let  $L : V \rightarrow V'$  be the elliptic operator

$$L = -\frac{d}{dx} \left( \alpha(x) \frac{d}{dx} \right)$$

where  $\alpha \in L^\infty(\Omega)$ ,  $\alpha(x) > 0$ . Let  $u$  be the solution of the following problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u(0) = u(1) = 0. \end{cases} \quad (1)$$

$\Rightarrow$  Oscillatory or jumping coefficients are allowed.

We need a small scale  $h$  to take care of the oscillatory coefficient.

## Fine grid discretisation $\mathcal{T}_h$

The  $P^1$ -Lagrange FE space  $V_h = \text{span}\{b_1^h, \dots, b_{N-1}^h\}$ .

Let  $P_h$  be the following isomorphism

$$P_h : \mathbb{R}^{N-1} \longrightarrow V_h \subset V$$
$$v = (v_1, \dots, v_{N-1}) \longmapsto P_h v = \sum_{i=1}^{N-1} v_i b_i^h.$$

Its adjoint is  $R_h := P_h^* \in L(V', \mathbb{R}^{N-1})$ .

The mass matrix:  $M_h := R_h P_h \in \mathbb{R}^{(N-1) \times (N-1)}$ .

The FE stiffness matrix  $L_h$  is given by

$$L_h = R_h L P_h.$$

## Coarse grid discretisation $\mathcal{T}_H$ , $H \gg h$

Let  $V_H$  be the linear FE space  $V_H = \text{span}\{b_1^H, \dots, b_{M-1}^H\}$ .  
The grid  $\mathcal{T}_H$  is nested in  $\mathcal{T}_h$  in the sense that  $V_H \subset V_h$ .

Let  $P_H$  be the following isomorphism

$$P_H : \mathbb{R}^{M-1} \rightarrow V_H \subset V$$

$$v = (v_1, \dots, v_{M-1}) \mapsto P_H v = \sum_{i=1}^{M-1} v_i b_i^H.$$

Its adjoint is  $R_H := P_H^* \in L(V', \mathbb{R}^{M-1})$ .

The mass matrix:  $M_H := R_H P_H \in \mathbb{R}^{(M-1) \times (M-1)}$ .

Let  $\|\cdot\|_2$  be the Euclidean norm and  $\|\!\| \cdot \|\!\|$  the norm defined for a matrix  $X \in \mathbb{R}^{(M-1) \times (M-1)}$  by

$$\|\!\| X^{-1} \|\!\| := \|P_H X^{-1} R_H\|_{L^2(\Omega) \leftarrow L^2(\Omega)} = \|M_H^{1/2} X^{-1} M_H^{1/2}\|_2.$$

## Notation

The inclusion  $V_H \subset V_h$  ensures that the following mappings are well defined:

1. the prolongation operator  $P_{h \leftarrow H}$  from the coarse space  $V_H$  to the fine space  $V_h$  given by

$$P_{h \leftarrow H} = (P_h^{-1} P_H) : \mathbb{R}^{M-1} \longrightarrow \mathbb{R}^{N-1} .$$

2. the restriction operator  $R_{H \leftarrow h} = P_{h \leftarrow H}^*$  from the fine space  $V_h$  to the coarse space  $V_H$ .

3. the normalised prolongation and restriction operators

$$\tilde{P}_{h \leftarrow H} = M_h P_{h \leftarrow H} M_H^{-1} \quad \text{and} \quad \tilde{R}_{H \leftarrow h} = \tilde{P}_{h \leftarrow H}^* .$$

1. Introduction
2. **Multiscale approach**
3. Theory for periodic problems
4. Numerical Experiment

## Our approach

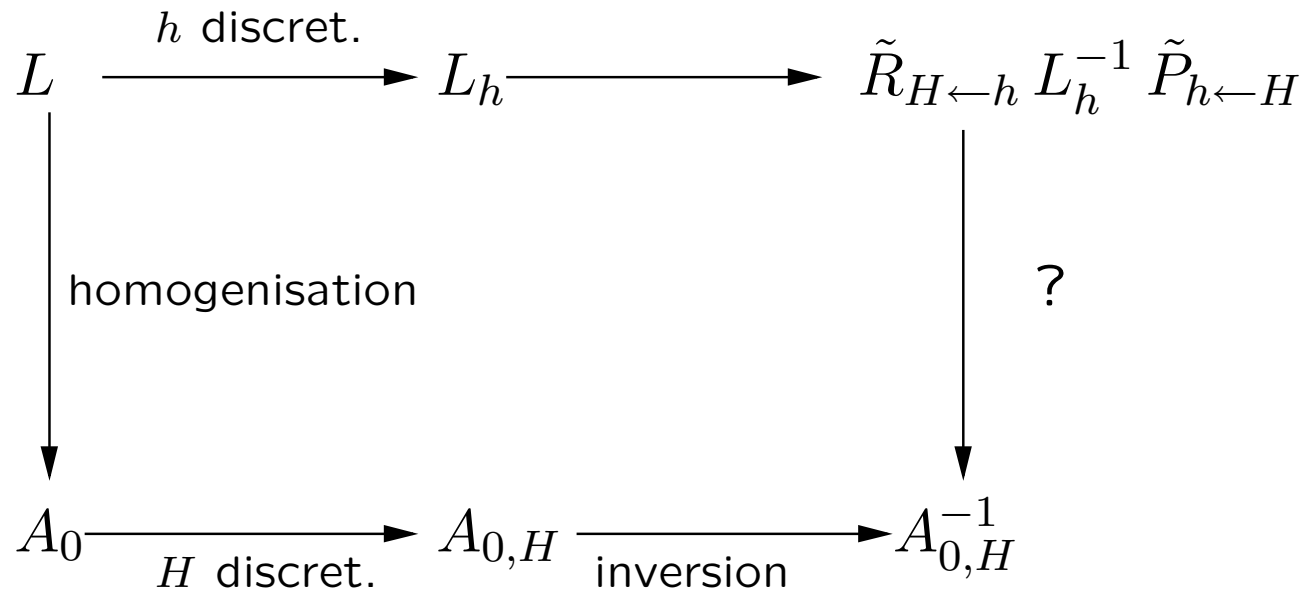
We consider the computation of  $\mathcal{L}_H$

$$\mathcal{L}_H := \left( \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} \right)^{-1}$$

**Question:** Does it exist an elliptic operator  $\mathcal{L}$ , which discretisation is  $\mathcal{L}_H$  on the coarse grid  $\mathcal{T}_H$ ?

**Remarks:**

- No requirement of regularity or periodicity of the coefficient.
- Available in any dimensions.



We consider the following problem:

We are looking for an elliptic operator  $A \in L(V, V')$  such that the inverse of its discretisation  $A_H^{-1}$  on the coarse grid satisfies:

$$A_H^{-1} \approx \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H}.$$

**Hope**  $A = -a \frac{d^2}{dx^2}$ ,  $a$  smoother than  $\alpha$ .

1. Introduction
2. Multiscale approach
3. **Theory for periodic problems**
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## Homogenisation

Let us consider the special case of a  $T$ -periodic coefficient  $\alpha$ .

### Homogenisation theory:

the exact solution  $u = L^{-1}f$  is approximated by the homogenised one  $u_0 = L_0^{-1}f$  with a precision depending on the period  $T$ .

The homogenised operator  $L_0$  associated with  $L$  is defined by

$$L_0 = -\alpha_0 \frac{d^2}{dx^2} \quad \text{with } \alpha_0 = \frac{1}{M\left(\frac{1}{\alpha}\right)} \quad \text{and} \quad M\left(\frac{1}{\alpha}\right) = \frac{1}{T} \int_T \frac{dx}{\alpha(x)}.$$

It is reasonable to expect that  $L_0$  a good solution of our problem, i.e.:

$$L_{0,H}^{-1} \approx \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H}.$$

## Theoretical result

Let  $L_{0,H}$  be the coarse FE-discretisation of  $L_0$ .

Comparison of  $\tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H}$  and  $L_{0,H}^{-1}$  defined on the coarse level.

**Theorem** *The following error estimate holds*

$$\| \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} - L_{0,H}^{-1} \| \leq C \left( \varepsilon(h) + T M \left( \frac{1}{\alpha} \right) (1 + 2T) + \varepsilon_0(H) \right),$$

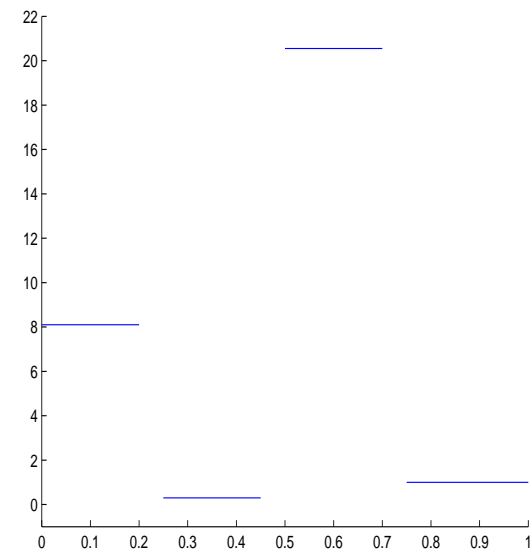
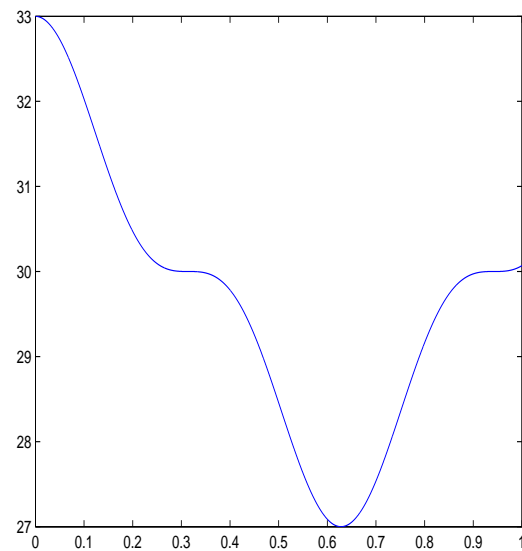
where  $\varepsilon(h)$  is a bound on the FE-discretisation error of  $L$  on the fine mesh  $\mathcal{T}_h$

$\varepsilon_0(H)$  is the bound on the FE-discretisation error of  $L_0$  on the coarse mesh  $\mathcal{T}_H$ .

1. Introduction
2. Multiscale approach
3. Theory for periodic problems
4. **Numerical Experiment**

## Numerical experiments

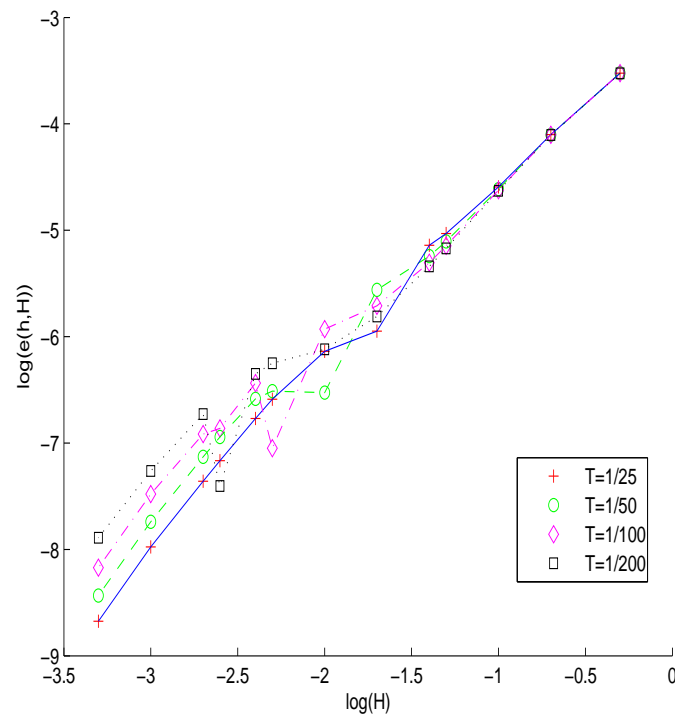
1.  $T$ -periodic coefficients
2. Non-periodic coefficients



## Smooth coefficient, small amplitude

Computation of the norm  $\| \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} - L_{0,H}^{-1} \|$  for:

- small step size:  $h=1/4000$
- large step size:  $H$  between  $1/2 = 2000h$  and  $2h$
- periods  $T = 1/25, 1/50, 1/100, 1/200$ .

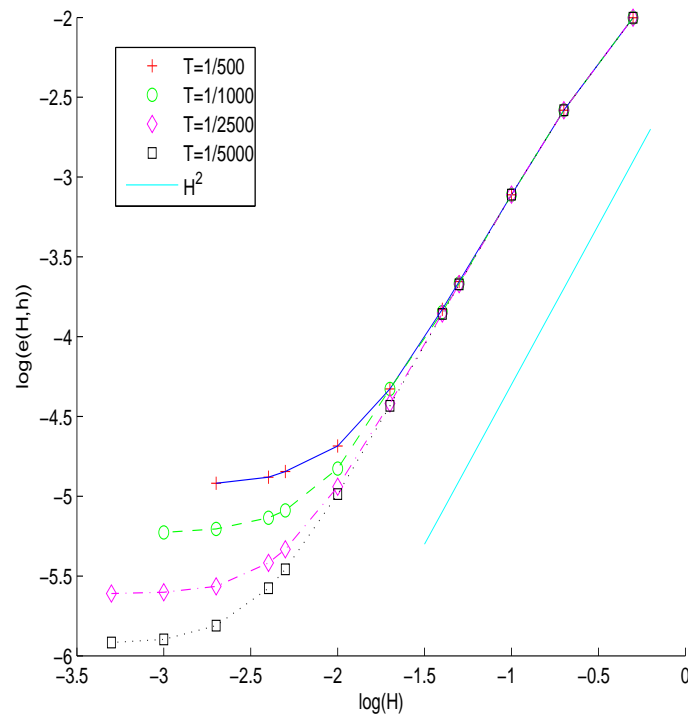


The convergence rate of  
is of order 1.5 with a  
jump around  $H = T/2$ .

## Piecewise constant coefficient

Computation of the norm  $\| \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} - L_{0,H}^{-1} \|$  for:

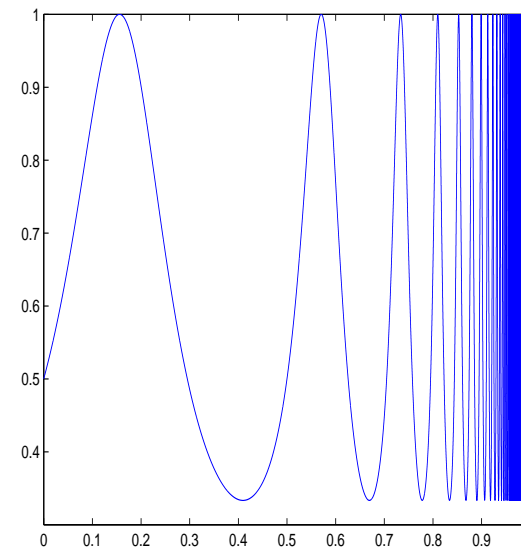
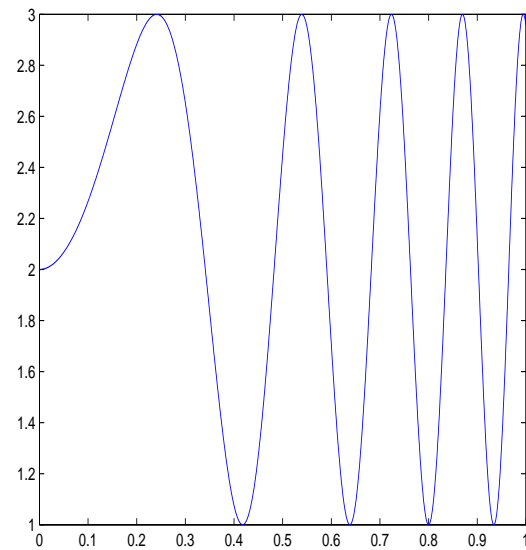
- small step size:  $h = 10^{-5}$
- large step size:  $H$  between  $1/2$  and  $50h$
- periods  $T = 1/500, 1/1000, 1/2500, 1/5000$ .



- plateau when  $T$  and  $H$  close to each other
- convergence of the norm when  $T \rightarrow 0$

## Numerical experiments

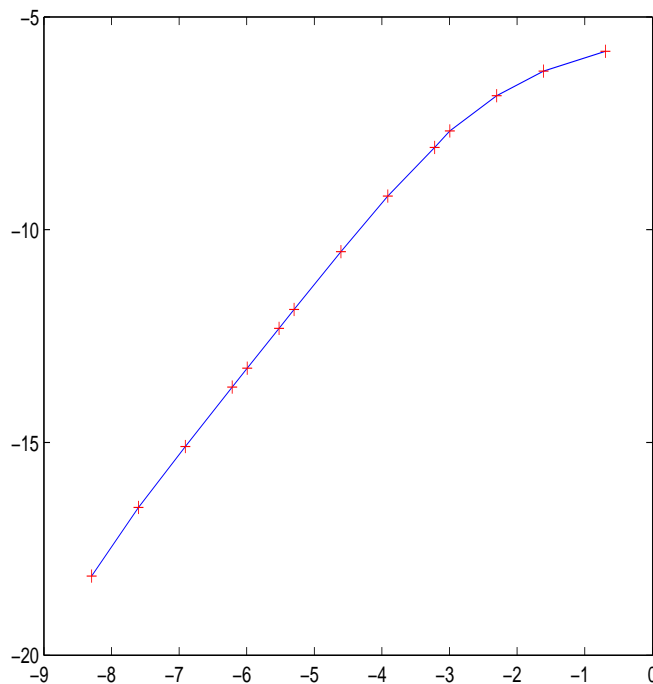
1.  $T$ -periodic coefficients
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**A smooth coefficient**  $\alpha(x) = 2 + \sin(27x^2)$ 

Computation of the norm  $\| \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} - L_{0,H}^{-1} \|$  for :

- the small step size  $h$  equals to  $h = 1/1000, 1/2000, 1/4000, 1/8000,$
- the coarse step size  $H$  ranges over  $1/2$  to  $1/4000$ .



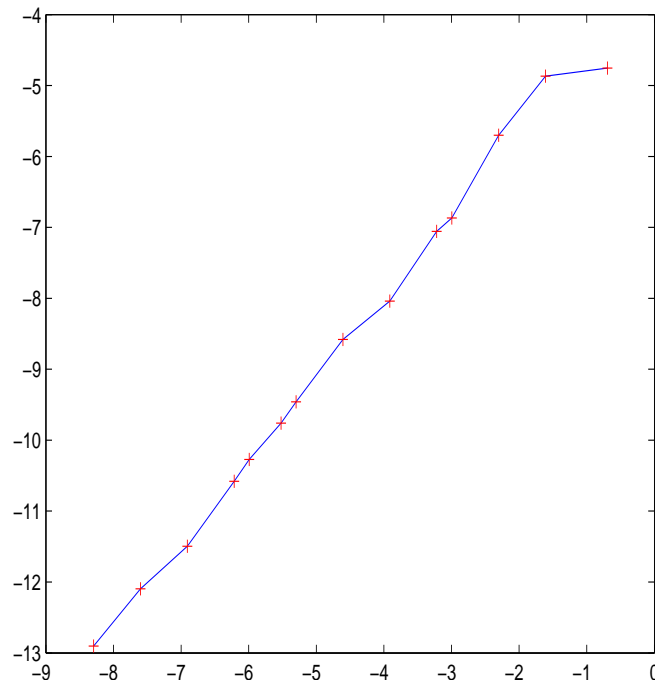
- Independence on the choice of the step size  $h$ .
  - Good convergence almost order  $\mathcal{O}(H^2)$  for  $H < 1/25$ .
- ⇒ Efficiency on coarse meshes.

## An oscillating coefficient

This function contains a continuum of scales.

Computation of the norm  $\| \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} - L_{0,H}^{-1} \|$  for :

- the small step size  $h$  equals to  $h = 1/1000, 1/2000, 1/4000, 1/8000,$
- the coarse step size  $H$  ranges over  $1/2$  to  $1/4000$ .



- Dependence on the choice of the step size  $h$  is small.
- Good convergence almost order  $\mathcal{O}(H)$ .

## Conclusion

- The choice of the operator  $A_0$  defined by  $A_0 = -a_0 \frac{d^2}{dx^2}$  satisfies

$$A_{0,H}^{-1} \approx \tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} .$$

- The method is efficient and robust.
- No smoothness or periodicity requirement of the coefficient  $\alpha$ .

## Future work

- Generalisation to 2D.
- Difficulty since the Green function is not analytically available.