Higher Order Entropies

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Higher order entropies (1)

- **A Thermodynamic interpretation of Bernstein Method**

Consider the heat equation

$$\partial_t u - \Delta u = 0$$

Defining

$$\zeta = |\partial_x u|^2 = \partial_x u{\cdot}\partial_x u$$

we have

$$\partial_t \zeta - \Delta\zeta + 2|\partial_x^2 u|^2 = 0$$

where $|\partial_x^2 u|^2 = \partial_x^2 u{:}\partial_x^2 u$ and ζ plays the role of an entropy even though we also have

$$\partial_t u^2 - \Delta u^2 + 2|\partial_x u|^2 = 0$$

For more general equations

$$\zeta = u^2 + c_1|\partial_x u|^2$$

Higher order entropies (2)

- **Enskog second order expansion of kinetic entropy**

The distribution function $f(t,x,c)$ of a monatomic gas is governed by Boltzmann equation

The kinetic entropy $S^{\rm kin}$ obeys the H theorem

$$S^{\rm kin} = -k_{\rm B} \int_{\mathbb{R}^n} f\big(\log f - 1\big) dc$$

Enskog expansion of f

$$f = f^{(0)}\big(1 + \varepsilon\phi^{(1)} + \mathcal{O}(\varepsilon^2)\big)$$

$$\begin{cases} f^{(0)} & \text{is the Maxwellian distribution} \\ \phi^{(1)} & \text{is the perturbation associated with the Navier-Stokes regime} \end{cases}$$

Expansion of kinetic entropy $S^{\rm kin}$

$$S^{\rm kin} = S^{(0)} + \varepsilon^2 S^{(2)} + \mathcal{O}(\varepsilon^3)$$

Higher order entropies (3)

- **Enskog second order kinetic entropy corrector**

Fluid entropy $S^{(0)}$

$$S^{(0)} = -k_{\mathrm{B}} \int_{\mathbb{R}^n} f^{(0)} \big(\log f^{(0)} - 1\big) dc$$

Enskog second order corrector $S^{(2)}$

$$S^{(2)} = -\frac{k_{\mathrm{B}}}{2} \int_{\mathbb{R}^n} f^{(0)} (\phi^{(1)})^2 dc$$

Explicit evaluation

$$S^{(2)} = -\frac{1}{\rho} \big(\overline{\lambda} |\partial_x T|^2 + \tfrac{1}{2} \overline{\eta} \, |d|^2 \, \big)$$

where

$$d = \partial_x v + \partial_x v^t - \tfrac{2}{n} (\partial_x \!\cdot v) \, I$$

$$\overline{\lambda} = (1/2rc_p)\lambda^2/T^3 \qquad \overline{\eta} = (1/4r)\eta^2/T^2$$

λ is the thermal conductivity, η the shear viscosity, and $c_p - c_v = r$

Higher order entropies (4)

- **Enskog higher order expansion of kinetic entropy**

Higher order expansion of f

$$f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \cdots + \epsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$$

Higher order expansion of S^{kin}

$$\mathcal{S}^{\mathrm{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$$

Structure of $S^{(2k)}$ in the absence of external forces

$$S^{(2k)} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \sum_{\nu} c_\nu \prod_{1\leq|\alpha|\leq 2k} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \Big(\frac{\partial^\alpha \rho}{\rho}\Big)^{\nu'_\alpha} \Big(\frac{\partial^\alpha v}{\sqrt{rT}}\Big)^{\nu''_\alpha}$$

where η is the shear viscosity, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq 2k}$, and

$$\sum_{1\leq|\alpha|\leq 2k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = 2k$$

The coefficients $c_{\alpha\beta\delta}$ are smooth bounded functions of $\log T$

Higher order entropies (5)

- **Kinetic entropy deviation estimators**

  Integration by parts in $\int_{\mathbb{R}^n} S^{(2k)}\, dx$ to eliminate derivatives of order stricly greater than k

  Use of classical interpolation inequalities

$$\Big| \int_{\mathbb{R}^n} S^{(2k)}\, dx \Big| \leq C \int_{\mathbb{R}^n} \gamma^{[k]}\, dx$$

  $\gamma^{[k]}$ is a $(2k)^{\rm th}$ order kinetic entropy deviation estimator

$$\gamma^{[k]} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}} \Big)^{2k} \Big(\big|\frac{\partial^k T}{T}\big|^2 + \big|\frac{\partial^k v}{\sqrt{rT}}\big|^2 + \big|\frac{\partial^k \rho}{\rho}\big|^2 \Big)$$

  Other $(2k)^{\rm th}$ order kinetic entropy deviation estimator

$$\widetilde{\gamma}^{[k]} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}} \Big)^{2k} \big(|\partial^k \log T|^2 + |\partial^k (v/\sqrt{rT})|^2 + |\partial^k \log \rho|^2 \big)$$

Higher order entropies (6)

- **Kinetic entropy deviation estimators**

  Zeroth order estimator

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = \frac{E^{(0)}}{T_\infty} - S^{(0)}$$

  Kinetic entropy estimators of $(2k)^{\mathrm{th}}$ order

$$\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]}$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}$$

- **Parallel with scalar equation**

$$\zeta^{[k]} = u^2 + c_1|\partial^1 u|^2 + \cdots + c_k|\partial^k u|^2$$

Higher order entropies (7)

- **Enskog expansion of Fisher Information**

  Leads to the same higher order estimators

- **Natural temperature scaling**
- **Persistence of kinetic entropy**

  Higher order entropies are tools to estimate the solution of a fluid model

  Different point of view from extended thermodynamics or burnett type models

  Will require small Mach numbers, which is compatible with Enskog expansion from

$$\mathrm{Ma} = \mathrm{Kn}\,\mathrm{Re}$$

where

$$\begin{cases} \mathrm{Ma} = v/\bar{c} & \text{is the Mach number} \\ \mathrm{Kn} = l/L & \text{is the Knudsen number} \\ \mathrm{Re} = \rho v L/\eta & \text{is the Reynolds number} \end{cases}$$

Preliminary analysis (1)

- **Incompressible model**

$$\partial_x \cdot v = 0$$
$$\partial_t(\rho v) + \partial_x \cdot (\rho v \otimes v + pI) - \partial_x \cdot \big(\eta(T)\, d\big) = 0$$
$$\partial_t(\rho c_v T) + \partial_x \cdot (\rho c_v T v) - \partial_x \cdot \big(\lambda(T)\, \partial_x T\big) = \tfrac{1}{2}\eta(T) d{:}d$$

- **General assumptions**

$$v,\ T - T_\infty \in C\big([0,\bar t], H^l\big) \cap C^1\big([0,\bar t], H^{l-2}\big) \cap L^2\big([0,\bar t], H^{l+1}\big), \qquad l \geq [n/2] + 2$$

- **Momentum equation**

$$\partial_t(\rho v) = \mathbb{P}\big(\partial_x \cdot (-\rho v \otimes v + \eta(T) d)\big), \qquad \mathbb{P} = \mathbb{I} + R \otimes R$$

$$R = (R_1, \ldots, R_n)^t, \qquad R_i = (-\Delta)^{-1/2} \partial_i, \quad 1 \leq i \leq n$$

- **Pressure**

$$p = \sum_{1 \leq i,j \leq n} R_i R_j \big(\rho v_i v_j - \eta d_{ij}\big)$$

Preliminary analysis (2)

- **Simplifying assumptions**

$$\lambda = \text{Cte}, \qquad \eta = \text{Cte}, \qquad c_v = \text{Cte}$$

- **Pressure**

$$p = \sum_{1\leq i,j\leq n} R_i R_j \big(\rho v_i v_j\big)$$

- **Second order entropy corrector**

Specialize general formulas to the incompressible case

$$\gamma = \frac{\text{A}\lambda}{T^{1+a}} |\partial_x T|^2 + \tfrac{1}{2} \frac{\text{A}\eta}{T^a} |d|^2$$

Positive parameters at our disposal

$$\text{A}_\lambda > 0, \qquad \text{A}_\eta > 0, \qquad a > 0$$

Note that v scales as $\sqrt{T}$ in γ, and that γ and $\gamma^{[1]}$ have similar properties

We will assume that $0 < a \leq 1$ in the following

Preliminary analysis (3)

- **Balance equation for γ**

$$\partial_t \gamma + \partial_x \cdot (v\gamma) + \partial_x \cdot \varphi + \pi + \Sigma + \omega = 0$$

where

$$\pi = \frac{2\mathrm{A}_\lambda \lambda}{\rho c_v} \frac{|\partial_x^2 T|^2}{T^{1+a}} + \frac{(1+a)(2+a)\mathrm{A}_\lambda \lambda}{\rho c_v} \frac{|\partial_x T|^4}{T^{3+a}}$$
$$+ \Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v} \Big) \frac{|d|^2\, |\partial_x T|^2}{T^{2+a}} + \frac{a\mathrm{A}_\eta \eta}{4\rho c_v} \frac{|d|^4}{T^{1+a}} + \frac{\mathrm{A}_\eta \eta}{\rho} \frac{|\partial_x d|^2}{T^a}$$

$$\Sigma = -\frac{4(1+a)\mathrm{A}_\lambda \lambda}{\rho c_v} \frac{\partial_x^2 T : \partial_x T \otimes \partial_x T}{T^{2+a}} - \Big(\frac{2\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a\mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta \eta}{\rho} \Big) \frac{\partial_x d \vdots d \otimes \partial_x T}{T^{1+a}}$$

$$\omega = \mathrm{A}_\lambda \frac{d : \partial_x T \otimes \partial_x T}{T^{1+a}} + 2 \frac{\mathrm{A}_\eta}{\rho} \frac{d : \partial_x^2 p}{T^a} + 2\mathrm{A}_\eta \frac{d : (\partial_x v \cdot \partial_x v)}{T^a}$$

$$\varphi = \frac{a\mathrm{A}_\eta \lambda}{2\rho c_v} \frac{|d|^2\, \partial_x T}{T^{1+a}} - \frac{\mathrm{A}_\eta \eta}{\rho} \frac{d : \partial_x d}{T^a} + \frac{(1+a)\mathrm{A}_\lambda \lambda}{\rho c_v} \frac{|\partial_x T|^2\, \partial_x T}{T^{2+a}} - \frac{2\mathrm{A}_\lambda \lambda}{\rho c_v} \frac{\partial_x^2 T \cdot \partial_x T}{T^{1+a}}$$

For a, b vectors, c, d matrices, $a \otimes b$ is the matrix with elements $a_i b_j$, $1 \le i, j \le n$, $c \otimes b$ is the third order tensor $c_{ij} b_k$, $1 \le i, j, k \le n$; $c : d$ is the quantity $\sum_{ij} c_{ij} d_{ij}$ and $|c|^2 = c : c$
For e and f third order tensors $e \vdots f$ denotes the quantity $\sum_{ijk} e_{ijk} f_{ijk}$ and $|e|^2 = e \vdots e$

Preliminary analysis (4)

- **Unconditional entropicity**

  Integrating the γ balance equation we obtain that

$$\partial_t \int_{\mathbb{R}^n} \gamma \, dx + \int_{\mathbb{R}^n} (\pi + \Sigma) \, dx = - \int_{\mathbb{R}^n} \omega \, dx$$

  We will say that unconditional entropicity holds if there exists $c > 0$ such that

$$\frac{1}{c} \int_{\mathbb{R}^n} \pi \, dx \leq \int_{\mathbb{R}^n} (\pi + \Sigma) \, dx \leq c \int_{\mathbb{R}^n} \pi \, dx$$

  for any $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, with $T \geq T_{\min} > 0$ and in this situation

$$\partial_t \int_{\mathbb{R}^n} \gamma \, dx + \frac{1}{c} \int_{\mathbb{R}^n} \pi \, dx \leq \int_{\mathbb{R}^n} |\omega| \, dx$$

- **Proposition :** Assume that the parameter a associated with γ is such that

$$0 < a < \inf\Big(\frac{4n-1}{2n^2+1}, 2\big(\frac{\lambda}{\eta c_v} + \frac{\eta c_v}{\lambda} \big)^{-1} \Big)$$

Then there exists positive constants A_λ and A_η such that unconditional entropicity holds. On the other hand, unconditional entropicity does not hold when a is close to unity.

Preliminary analysis (5)

- **Sketch of the Proof**

  Explicit evaluation of

$$\int_{\mathbb{R}^n} (\pi + \Sigma)\, dx$$

  Integration by parts and polar decomposition of the Hessian matrix $\partial_x^2 T$

$$\partial_x^2 T = \widehat{\partial_x^2 T} + (\Delta T/n)\mathbb{I}$$

  Binomial formula

  Counter exemple for $a = 1$

- **Inequalities**

  One can establish that for $a \neq -2$, $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $T \geq T_{\min} > 0$

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}}\, dx \leq c \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}}\, dx$$

  Different inequalities will be established with powers of $\| \log T \|_{BMO}$ in the right hand side

Preliminary analysis (6)

- **Expression of $\partial_x v$ in terms of d**

  For any $v \in H^1$ and any index pair (i,j) we have

$$2\partial_j v_i = d_{ij} - \sum_{1\le l\le n} R_l R_j d_{li} + \sum_{1\le l\le n} R_l R_i d_{lj}$$

  and we also have

$$2\partial_j \partial_k v_i = \partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}$$

- **Estimates of convective terms**

  Assume that $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \ge T_{\min} > 0$, and $a \le 1/3$. Then

$$\int_{\mathbb{R}^n} |\omega|\, dx \le c\Big(\int_{\mathbb{R}^n} \pi\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} |d|^2\, dx\Big)^{1/2} \sup_{\mathbb{R}^n} T^{(1-a)/2}$$

  We also have

$$\int_{\mathbb{R}^n} |\omega|\, dx \le c\Big(\int_{\mathbb{R}^n} \pi\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} \gamma\, dx\Big)^{1/2} \sup_{\mathbb{R}^n} T^{1/2}$$

Preliminary analysis (7)

- **Temperature weights**

  Nonlinear convective terms introduce a multiplication by the velocity field v

  Since v scales as $\sqrt{T}$ we recover the $\sqrt{T}$ weight of the convective terms

  The $\sup_{\mathbb{R}^n} T$ factors cannot be controled by $\int_{\mathbb{R}^n} \pi \, dx$ and $\int_{\mathbb{R}^n} \gamma \, dx$

  Small values of a yields unconditional entropicity but prevent majorization of convective terms

  Larger values of a yields majorization of convective terms but prevent unconditional entropicity

- **Partial conclusion**

  Temperature dependence of transport coefficients λ and η

  Conditional entropicity properties

- **Remarks**

  Same conclusions for periodic domains

  Assuming $\partial_x T/T \in L^2 \cap L^4$ when $n = 3$ implies that $\log T$ has a finite limit at infinity and $T_\infty > 0$

Weighted inequalities (0)

- **Differential indentities**

Letting $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we have

$$\partial^\alpha (fg) = \sum_{0 \le \beta \le \alpha} c_{\alpha\beta}\, \partial^\beta f \partial^{(\alpha-\beta)} g$$

For f and g smooth scalar functions we have

$$\partial^\alpha (g \circ f) = \sum_{\sigma\mu} c_{\sigma\mu}\, \partial^\sigma g \prod_{1 \le |\beta| \le |\alpha|} \big(\partial^\beta f\big)^{\mu_\beta}$$

where $c_{\sigma\mu}$ are nonnegative integer coefficients

The sum is over $1 \le \sigma \le |\alpha|$, $\mu = (\mu_\beta)_{1 \le |\beta| \le |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, such that

$$\sum_{1 \le |\beta| \le |\alpha|} \mu_\beta = \sigma, \qquad \sum_{1 \le |\beta| \le |\alpha|} \beta\, \mu_\beta = \alpha$$

so that we have in particular $\sum_\beta |\beta|\, \mu_\beta = |\alpha|$.

Weighted inequalities (1)

- **A_p class**

  For $1 < p < \infty$, $g \in L^1_{loc}(\mathbb{R}^n)$, positive and locally integrable , g is in the A_p class if

$$[g]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q g\, dx \right) \left(\frac{1}{|Q|} \int_Q g^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty$$

  We have $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithm in BMO

- **BMO space**

  A locally summable function f belongs to the space $BMO(\mathbb{R}^n)$ when

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q \left| f(x) - \bar{f}_Q \right| dx < \infty$$

- **Fine properties of BMO**

  There exists constants $b(n)$ and $B(n)$ such that for any $\theta \in \mathbb{R}$, $u \in BMO$, and $1 < p < \infty$

$$|\theta| \|u\|_{BMO} < b(n)/2, \qquad |\theta| \|u\|_{BMO} < (p-1)b(n)/2$$

  implies that $\exp(\theta u) \in A_p$ and $\left[\exp(\theta u)\right]_{A_p} \le \left(1 + B(n)\right)^p$

Weighted inequalities (2)

- **Weighted Calderón-Zygmund operators**

  Let $\mathcal{G}$ be a Calderón-Zygmund operator, $1 < p < \infty$, and g be a weight in A_p

  Then $\mathcal{G}$ is bounded in $L^p\big(gdx\big)$, or equivalently, $g^{1/p}\mathcal{G}g^{-1/p}$ is bounded in L^p

  with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p, [g]_{A_p})$

- **Calderón-Zygmund operators**

Let $\mathcal{G} : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator and K the restriction of the distribution kernel associated with $\mathcal{G}$ to the open set $x \neq y$ of $\mathbb{R}^n \times \mathbb{R}^n$. We say that $\mathcal{G}$ is a Calderón-Zygmund operator when

(i) K is a locally integrable function and there exists c_0 such that for $x \neq y$ $|K(x, y)| \leq c_0|x - y|^{-n}$

(ii) There exists $\delta \in (0, 1]$ and c_1 such that for $x \neq y$ and $|x' - x| \leq \frac{1}{2}|x - y|$ we have
$|K(x', y) - K(x, y)| \leq c_1|x' - x|^\delta |x - y|^{-n-\delta}$

(iii) Similarly if $x \neq y$ and $|y' - y| \leq \frac{1}{2}|x - y|$ we have $|K(x, y') - K(x, y)| \leq c_1|y' - y|^\delta |x - y|^{-n-\delta}$

(iv) $\mathcal{G}$ can be extended into a continuous linear operator over $L^2(\mathbb{R}^n)$ with norm lower or equal to c_2

Weighted inequalities (3)

- **Weighted multilinear estimates**

Let k, l be positive integers, and α^j, $1 \leq j \leq l$, be multiindices such that $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $k = \sum_{1\leq j\leq l} |\alpha^j|$. Let $1 < p < \infty$, $g \in A_p$, and $u_1, \ldots, u_l$, be such that there exist constants $u_{j,\infty}$ with $u_j - u_{j,\infty} \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $g^{\frac{1}{p}} \partial^k u_j \in L^p$, $1 \leq j \leq l$. There exists $c = c(k, n, p, [g]_{A_p})$ such that

$$\|g^{1/p} \prod_{1\leq j\leq l} \partial^{\alpha^j} u_j\|_{L^p} \leq c\Big(\sum_{1\leq j\leq l} \|u_j\|_{BMO}\Big)^{l-1}\Big(\sum_{1\leq j\leq l} \|g^{1/p}\partial^k u_j\|_{L^p}\Big)$$

where we define

$$\|g^{1/p}\partial^k \mathsf{v}\|_{L^p}^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\mathbb{R}^n} g|\partial^\alpha \mathsf{v}|^p \, dx$$

- **Sketch of the Proof**

  Fourier transform and density of $\mathcal{D}(\mathbb{R}^n)$ in $H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$

  Coifman Meyer theory of multilinear operators

  Continuity of weighted Calderón-Zygmund operators

- **Remark**

  $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ if and only if $k \geq n/2$

Weighted inequalities (4)

- **Weighted multilinear estimates for τ and w**

Let $k \geq 1$, $\bar{\theta} > 0$, $1 < p < \infty$, τ such that $\tau - \tau_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ for some constant τ_∞. There exist positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$ such that if $\|\tau\|_{BMO} < \delta$, then for any $|\theta| \leq \bar{\theta}$, any integer $l \geq 1$, and any multiindices α^j, $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1 \leq j \leq l} |\alpha^j| = k$, whenenver $\exp(\theta\tau/p)\partial^k\tau \in L^p(\mathbb{R}^n)$, we have

$$\Big\| e^{\frac{\theta\tau}{p}} \prod_{1 \leq j \leq l} (\partial^{\alpha^j}\tau) \Big\|_{L^p} \leq c \|\tau\|_{BMO}^{l-1} \, \|e^{\frac{\theta\tau}{p}} \partial^k \tau\|_{L^p}.$$

Further assuming that $w \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $e^{\theta\tau/p}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \leq \bar{l} \leq l$, we have

$$\Big\| e^{\frac{\theta\tau}{p}} \prod_{1 \leq j \leq \bar{l}} (\partial^{\alpha^j}\tau) \prod_{\bar{l}+1 \leq j \leq l} (\partial^{\alpha^j} w) \Big\|_{L^p} \leq c \Big(\|\tau\|_{BMO} + \|w\|_{BMO} \Big)^{l-1} \Big(\|e^{\frac{\theta\tau}{p}} \partial^k \tau\|_{L^p} + \|e^{\frac{\theta\tau}{p}} \partial^k w\|_{L^p} \Big)$$

where, in the left hand side, we have denoted by w any of its components $w_1, \ldots, w_n$

- **Sketch of the Proof**

Weighted multilinear estimates applied to τ and w and properties of BMO

Weighted inequalities (5)

- **Auxiliary functions**

$$\tau = \log T \qquad w = \frac{v}{\sqrt{T}}$$

- **Derivatives of auxiliary functions**

For T smooth, positive and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we have

$$\frac{\partial^\alpha T}{T} = \partial^\alpha \tau + \sum_{\mu} c_\mu \prod_{1 \le |\beta| \le |\alpha|-1} (\partial^\beta \tau)^{\mu_\beta}$$

where $\mu = (\mu_\beta)_{1 \le |\beta| \le |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, c_μ are nonnegative integer coefficients, and

$$\sum_{1 \le |\beta| \le |\alpha|} \beta \, \mu_\beta = \alpha$$

so that we have in particular $\sum_\beta |\beta| \, \mu_\beta = |\alpha|$. Similarly, we have for $1 \le i \le n$

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu \tilde{\alpha}} c_{\mu \tilde{\alpha}} \prod_{1 \le |\beta| \le |\alpha|} (\partial^\beta \tau)^{\mu_\beta} \, \partial^{\tilde{\alpha}} w_i \, + \sum_{\mu} c_{\mu 0} \prod_{1 \le |\beta| \le |\alpha|} (\partial^\beta \tau)^{\mu_\beta} \, w_i,$$

with $0 \le \tilde{\alpha} \le \alpha$ and $\sum_{1 \le |\beta| \le |\alpha|} \beta \, \mu_\beta + \tilde{\alpha} = \alpha$

Weighted inequalities (6)

- **Weighted multilinear estimates for T and v**

Let $k \geq 1$, $\bar{\theta} > 0$, $1 < p < \infty$, T with $T \geq T_{\min} > 0$ and $T - T_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ where $T_\infty > 0$. There exist positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$ such that if $\| \log T\|_{BMO} < \delta$, then for any $|\theta| \leq \bar{\theta}$, any integer $l \geq 1$, and any multiindices α^j, $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1 \leq j \leq l} |\alpha^j| = k$, whenenver $T^{\theta/p}(\partial^k T)/T \in L^p(\mathbb{R}^n)$, we have

$$\Big\|T^{\frac{\theta}{p}} \prod_{1\leq j\leq l} \Big(\frac{\partial^{\alpha^j} T}{T}\Big)\Big\|_{L^p} \leq c\big\|\log T\big\|_{BMO}^{l-1} \; \|T^{\frac{\theta}{p}} \frac{\partial^k T}{T}\|_{L^p}.$$

Further assuming $v \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $\| \log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$, whenever $T^{\theta/p}(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$, we have for $0 \leq \bar{l} \leq l$

$$\Big\|T^{\frac{\theta}{p}} \prod_{1\leq j\leq \bar{l}} \Big(\frac{\partial^{\alpha^j} T}{T}\Big) \prod_{\bar{l}+1\leq j\leq l} \Big(\frac{\partial^{\alpha^j} v}{\sqrt{T}}\Big)\Big\|_{L^p} \leq c\Big(\big\|\log T\big\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty}\Big)^{l-1} \Big(\|T^{\frac{\theta}{p}} \frac{\partial^k T}{T}\|_{L^p} + \|T^{\frac{\theta}{p}} \frac{\partial^k v}{\sqrt{T}}\|_{L^p}\Big)$$

where, in the left hand member, we have denoted by v any of its components $v_1, \dots, v_n$.

- **Sketch of the Proof**

Weighted multilinear estimates are applied to τ and w and transported to T and v

Weighted inequalities (7)

- **Weighted interpolation inequality**

Let $1 < q < \infty$, $1 < r < \infty$, $k \geq 1$, $0 \leq j \leq k$, $g \in A_r \cap A_q = A_{\min(q,r)}$ and let p be such that

$$\frac{1}{p} = \frac{k-j}{k}\frac{1}{q} + \frac{j}{k}\frac{1}{r}$$

For $\mathsf{v} \in L^q(gdx)$ and $\partial^k \mathsf{v} \in L^r(gdx)$, we have $\partial^j \mathsf{v} \in L^p(gdx)$ and there exists $\mathcal{C}(n, k, q, r, [g]_{A_q}, [g]_{A_r})$ with

$$\Big(\int_{\mathbb{R}^n} g|\partial^j \mathsf{v}|^p \, dx\Big)^{\frac{1}{p}} \leq \mathcal{C}\Big(\int_{\mathbb{R}^n} g|\mathsf{v}|^q \, dx\Big)^{(1-\frac{j}{k})\frac{1}{q}} \Big(\int_{\mathbb{R}^n} g|\partial^k \mathsf{v}|^r \, dx\Big)^{\frac{j}{k}\frac{1}{r}}.$$

- **Weighted interpolation of derivatives**

Let $1 < r < \infty$, $k \geq 1$, $1 \leq j \leq k$, $g \in A_r$ and $1/p = j/kr$. Then for v with $\mathsf{v} - \mathsf{v}_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $\partial^k \mathsf{v} \in L^r(gdx)$, we have $\partial^j \mathsf{v} \in L^p(gdx)$ and there exists $\mathcal{C}(n, k, r, [g]_{A_r})$ such that

$$\Big(\int_{\mathbb{R}^n} g|\partial^j \mathsf{v}|^{\frac{rk}{j}} \, dx\Big)^{\frac{j}{rk}} \leq \mathcal{C}\, \|\mathsf{v}\|_{BMO}^{1-\frac{j}{k}} \Big(\int_{\mathbb{R}^n} g|\partial^k \mathsf{v}|^r \, dx\Big)^{\frac{j}{rk}}$$

Weighted inequalities (8)

- **Sketch of the Proofs**

  (i) Induction on k and Grafakos and Torres theory of multilinear singular integral operators

  (ii) Letting $p = r$, $|\alpha^j| = 1$, and $k = l$ in the weighted estimates of Coifman Meyer type we obtain the inequality for $j = 1$. For $1 < j < k$ we can then interpolate $\partial^j \mathsf{v}$ between $\partial^1 \mathsf{v}$ and $\partial^k \mathsf{v}$

- **Remark**

  As a special case we obtain that for $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \geq T_{\min} > 0$ and $\| \log T \|_{BMO}$ small enough, we have

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}} \, dx \leq c \| \log T \|_{BMO}^2 \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}} \, dx$$

Higher order entropy estimates (1)

- **Incompressible model**

$$
\begin{aligned}
&\partial_x\cdot v = 0\\
&\partial_t(\rho v) + \partial_x\cdot(\rho v\otimes v + pI) - \partial_x\cdot\big(\eta(T)\,d\big) = 0\\
&\partial_t(\rho c_v T) + \partial_x\cdot(\rho c_v T v) - \partial_x\cdot\big(\lambda(T)\,\partial_x T\big) = \tfrac{1}{2}\eta(T) d{:}d
\end{aligned}
$$

- **General assumptions**

$$
v,\ T - T_\infty \in C\big([0,\bar t], H^l\big) \cap C^1\big([0,\bar t], H^{l-2}\big) \cap L^2\big([0,\bar t], H^{l+1}\big), \qquad l \geq [n/2] + 2
$$

- **Momentum equation**

$$
\partial_t(\rho v) = \mathbb{P}\big(\partial_x\cdot(-\rho v\otimes v + \eta(T) d)\big), \qquad \mathbb{P} = \mathbb{I} + R\otimes R
$$

$$
R = (R_1, \dots, R_n)^t, \qquad R_i = (-\Delta)^{-1/2}\partial_i, \quad 1 \leq i \leq n
$$

- **Pressure**

$$
p = \sum_{1\leq i,j\leq n} R_i R_j \big(\rho v_i v_j - \eta d_{ij}\big)
$$

Higher order entropy estimates (2)

- **Temperature dependent transport coefficients**

  There exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, $\overline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}}_\sigma > 0$ for any $\sigma \geq 1$ such that

$$\underline{\mathfrak{a}}\, T^{\varkappa} \leq \lambda / c_v \leq \overline{\mathfrak{a}}\, T^{\varkappa}, \qquad \underline{\mathfrak{a}}\, T^{\varkappa} \leq \eta \leq \overline{\mathfrak{a}}\, T^{\varkappa}, \qquad T^{\sigma}\big(|\partial_T^{\sigma} \lambda| + |\partial_T^{\sigma} \eta| \big) \leq \overline{\mathfrak{a}}_\sigma\, T^{\varkappa}$$

  Kinetic theory suggests $1/2 \leq \varkappa \leq 1$ but $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate

  We assume for the sake of simplicity that $c_v = \mathrm{Cte}$

- **Interaction potentials between particles**

  Power laws exactly valid for points centers of repulsion

  Power laws asymptotically valid for Lenard-Jones interaction potentials

  Assumptions also valid for polyatomic gases

- **Rescaling of solutions**

  In the special case $\lambda = \mathfrak{a}_\lambda T^{\varkappa}$, $\eta = \mathfrak{a}_\eta T^{\varkappa}$, $c_v = \mathrm{Cte}$, if $v(t,x)$ and $T(t,x)$ are a solution then

$$\xi\, v(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x), \qquad \xi^2\, T(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x)$$

  are also a solution for any positive ξ

Higher order entropy estimates (3)

- **Higher order entropies**

  Specialize general formulas to the incompressible case

$$\gamma^{[k]} = \mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{1+a_k}} + \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{a_k}}$$

$$|\partial^k T|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha T)^2, \qquad |\partial^k v|^2 = \sum_{1 \le i \le n} |\partial^k v_i|^2$$

  where $\mathrm{A}_\lambda^{[k]} > 0$, $\mathrm{A}_\eta^{[k]} > 0$, and $a_k \in \mathbb{R}$

- **Modified higher order entropies**

  Denoting $\tau = \log T$ and $w = v/\sqrt{T}$ we define

$$\tilde{\gamma}^{[k]} = \exp\big((1-a_k)\tau\big)\Big(\mathrm{A}_\lambda^{[k]} |\partial^k \tau|^2 + \mathrm{A}_\eta^{[k]} |\partial^k w|^2\Big)$$

  The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ will be shown to have similar properties

Higher order entropy estimates (4)

- **Zeroth order modified entropy**

  We define $\gamma^{[0]} = \tilde{\gamma}^{[0]}$, for $0 < a_0 \leq 1$, by

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = (\mathrm{A}_\lambda^{[0]} + \mathrm{A}_\eta^{[0]})\zeta^{[0]}$$

  where $\mathrm{A}_\lambda^{[0]} > 0$, $\mathrm{A}_\eta^{[0]} > 0$, and

$$\zeta^{[0]} = \begin{cases} \dfrac{T - T_\infty}{T_\infty} - \log\Big(\dfrac{T}{T_\infty}\Big) + \frac{1}{2}\dfrac{v^2}{c_v T_\infty}, & \text{if} \quad a_0 = 1 \\[2ex] \dfrac{T - T_\infty}{T_\infty^{a_0}} - \dfrac{T^{1-a_0} - T_\infty^{1-a_0}}{1 - a_0} + \frac{1}{2}\dfrac{v^2}{c_v T_\infty^{a_0}}, & \text{if} \quad 0 < a_0 < 1 \end{cases}$$

- **Higher order entropy estimators**

  We define the $(2k)^{\mathrm{th}}$ order kinetic entropy estimators by

$$\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]} \qquad k \geq 0$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]} \qquad k \geq 0$$

Higher order entropy estimates (5)

- **Governing equation for $\gamma^{[k]}$**

For $k \geq 1$ and (v, T) smooth solution of the incompressible Navier-Stokes equations we have

$$\partial_t \gamma^{[k]} + \partial_x \cdot (v \gamma^{[k]}) + \partial_x \cdot \varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0$$

where

$$\pi_\gamma^{[k]} = \frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v} \frac{|\partial^{k+1} T|^2}{T^{1+a_k}} + \frac{2\eta \mathrm{A}_\eta^{[k]}}{\rho} \frac{|\partial^{k+1} v|^2}{T^{a_k}}$$

$$\Sigma_\gamma^{[k]} = \sum_{\sigma\nu\mu} T^{\sigma-\varkappa} (c_{\sigma\nu\mu} \partial_T^\sigma \lambda + c'_{\sigma\nu\mu} \partial_T^\sigma \eta)\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu\mathcal{R}} c_{\sigma\nu\mu\mathcal{R}} \Pi_\nu^{(k+1)} \mathcal{R}\big(T^{\sigma-\varkappa} \partial_T^\sigma \eta\, \Pi_\mu^{(k+1)}\big)$$

where $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\mathcal{R}$ singular operator in the form $T^{-\theta} R_i R_j T^\theta$ with $\theta = (a_k + \varkappa)/2$, and

$$\Pi_\nu^{(k+1)} = T^{(1-a_k+\varkappa)/2} \prod_{1\leq|\alpha|\leq k+1} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \prod_{1\leq|\alpha|\leq k+1} \Big(\frac{\partial^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}$$

where v denotes any of its components $v_1, \ldots, v_n$, and μ and ν must be such that

$$\sum_{1\leq|\alpha|\leq k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = \sum_{1\leq|\alpha|\leq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leq 1$$

so that there is at most one derivative of $(k+1)^{\mathrm{th}}$ order in the product $\Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$.

Higher order entropy estimates (6)

- **Governing equation for $\gamma^{[k]}$ (continued)**

The term $\omega_\gamma^{[k]}$ is given by

$$\omega_\gamma^{[k]}\, T^{-(1-2\varkappa+a_{k-1}-a_k)/2} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu\mathcal{R}} c_{\nu\mu\mathcal{R}}\Pi_\nu^{(k)}\mathcal{R}\big(\Pi_\mu^{(k+1)}\big)$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha+\nu'_\alpha) = k, \qquad \sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha+\mu'_\alpha) = k+1$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha)=0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$ and the singular operator $\mathcal{R}$ is in the form $T^{-\theta}R_iR_jT^\theta$ with $\theta=(1+a_k-\varkappa)/2$ and $1\le i,j\le n$. Finally the flux $\varphi_\gamma^{[k]}=(\varphi_{\gamma 1}^{[k]},\dots,\varphi_{\gamma n}^{[k]})$ is in the form

$$\varphi_{\gamma l}^{[k]}T^{-\frac{a_{k-1}-a_k}{2}} = \sum_{\sigma\nu\mu l} T^{\sigma-\varkappa}(c_{\sigma\nu\mu l}\partial_T^\sigma\lambda + c'_{\sigma\nu\mu l}\partial_T^\sigma\eta)\ \Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu\mathcal{R}l} c_{\sigma\nu\mu\mathcal{R}l}\Pi_\nu^{(k)}\mathcal{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma\eta\ \Pi_\mu^{(k+1)}\big)$$

Higher order entropy estimates (7)

- **Governing equation for $\tilde{\gamma}^{[k]}$**

For $k \geq 1$ and (v, T) smooth solution of the incompressible Navier-Stokes equations we have

$$\partial_t \tilde{\gamma}^{[k]} + \partial_x \cdot (v \tilde{\gamma}^{[k]}) + \partial_x \cdot \varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0$$

where

$$\pi_{\tilde{\gamma}}^{[k]} = e^{(1-a_k)\tau} \Big(\frac{2\lambda \mathrm{A}_{\lambda}^{[k]}}{\rho c_v} \, |\partial^{k+1}\tau|^2 + \frac{2\eta \mathrm{A}_{\eta}^{[k]}}{\rho} |\partial^{k+1} w|^2 \Big)$$

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma\nu\mu} e^{-\varkappa\tau} (c_{\sigma\nu\mu} \partial_\tau^\sigma \lambda + c'_{\sigma\nu\mu} \partial_\tau^\sigma \eta) \; \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu\iota\mathcal{R}} c_{\sigma\nu\mu\iota\mathcal{R}} \Pi_\nu^{(k+1)} \Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)} + \widehat{\Sigma}_{\tilde{\gamma}}^{[k]}$$

where $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, with

$$\Pi_\nu^{(k+1)} = e^{(1-a_k+\varkappa)\tau/2} \prod_{1\leq|\alpha|\leq k+1} (\partial^\alpha \tau)^{\nu_\alpha} \prod_{0\leq|\alpha|\leq k+1} (\partial^\alpha w)^{\nu'_\alpha}$$

where w denotes any of its components $w_1, \ldots, w_n$ and μ and ν must be such that

$$\sum_{1\leq|\alpha|\leq k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = \sum_{1\leq|\alpha|\leq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leq 1$$

Higher order entropy estimates (8)

- **Governing equation for $\tilde{\gamma}^{[k]}$ (continued)**

The non strictly differential terms $\Pi^{(k+1)}_{\mu\iota\mathcal{R}\sigma}$ are given by

$$\Pi^{(k+1)}_{\mu\iota\mathcal{R}\sigma} = \widetilde{\Pi}^{(k+1,l)}_{\mu}\mathcal{R}\Big(e^{-\varkappa\tau}\partial^{\sigma}_{\tau}\eta\,\widetilde{\Pi}^{(k+1,k+1-l)}_{\iota}\Big)$$

with

$$\widetilde{\Pi}^{(k+1,l)}_{\mu} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}}\prod_{1\leq|\alpha|\leq k+1}\big(\partial^{\alpha}\tau\big)^{\mu_\alpha}\prod_{0\leq|\alpha|\leq k+1}\big(\partial^{\alpha}w\big)^{\mu'_\alpha}$$

$$\widetilde{\Pi}^{(k+1,k+1-l)}_{\iota} = e^{(1-a_k+\varkappa)\frac{(k+1-l)\tau}{2(k+1)}}\prod_{1\leq|\alpha|\leq k+1}\big(\partial^{\alpha}\tau\big)^{\iota_\alpha}\prod_{0\leq|\alpha|\leq k+1}\big(\partial^{\alpha}w\big)^{\iota'_\alpha}$$

where $\nu = (\nu_\alpha,\nu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha,\mu'_\alpha)_{1\leq|\alpha|\leq k}$, $\iota = (\iota_\alpha,\iota_\alpha)_{1\leq|\alpha|\leq k}$, $0\leq\sigma\leq k$, and

$$\sum_{1\leq|\alpha|\leq k+1}|\alpha|(\mu_\alpha+\mu'_\alpha) = l, \quad \sum_{1\leq|\alpha|\leq k+1}|\alpha|(\iota_\alpha+\iota'_\alpha) = k+1-l, \quad \sum_{|\alpha|=k+1}(\nu_\alpha+\nu'_\alpha+\mu_\alpha+\mu'_\alpha+\iota_\alpha+\iota'_\alpha)\leq 1$$

for $0\leq l\leq k$, $\mathcal{R} = e^{-\theta\tau}R_iR_je^{\theta\tau}$, $1\leq i,j\leq n$, with $2\theta = a_k+\varkappa+l(1-a_k+\varkappa)/(k+1)$ and we have

$$\widehat{\Sigma}^{[k]}_{\tilde{\gamma}} = \frac{\mathrm{A}^{[k]}_{\eta}}{\rho}\Big(\frac{\lambda}{c_v}-\eta\Big)e^{(1-a_k)\tau}\sum_{|\alpha|=k}\frac{k!}{\alpha!}w{\cdot}\partial^{\alpha}w\,\partial^{\alpha}\tau$$

Higher order entropy estimates (9)

- **Governing equation for $\tilde{\gamma}^{[k]}$ (continued)**

The term $\omega^{[k]}_{\tilde{\gamma}}$ is given by

$$\omega^{[k]}_{\tilde{\gamma}}\ e^{-(1-2\varkappa+a_{k-1}-a_k)\tau/2} = \sum_{\nu\mu} c_{\nu\mu}\Pi^{(k)}_{\nu}\Pi^{(k+1)}_{\mu} + \sum_{\nu\mu\iota\mathcal{R}} c_{\nu\mu\iota\mathcal{R}}\Pi^{(k)}_{\nu}\Pi^{(k+1)}_{\mu\iota\mathcal{R}}$$

where $\sum_{1\le|\alpha|\le k}\ |\alpha|(\nu_\alpha+\nu'_\alpha)=k$, $\sum_{1\le|\alpha|\le k}\ |\alpha|(\mu_\alpha+\mu'_\alpha)=k+1$, and $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha)=0$. For the non strictly differential terms

$$\Pi^{(k+1)}_{\mu l\mathcal{R}} = \widetilde{\Pi}^{(k+1,l)}_{\mu}\mathcal{R}\big(\widetilde{\Pi}^{(k+1,k+1-l)}_{\iota}\big)$$

we have

$$\sum_{1\le|\alpha|\le k}|\alpha|(\nu_\alpha+\nu'_\alpha)=k,\ \sum_{1\le|\alpha|\le k}|\alpha|(\mu_\alpha+\mu'_\alpha)=l,\ \sum_{1\le|\alpha|\le k}|\alpha|(\iota_\alpha+\iota'_\alpha)=k+1-l$$

and $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha+\iota_\alpha+\iota'_\alpha)=0$ for some $0\le l\le k$ and the singular operator $\mathcal{R}$ is in the form $T^{-\theta}R_iR_jT^{\theta}$ with $1\le i,j\le n$ and $2(k+1)\theta = 2(k+1)-(k+1-l)(1-a_k+\varkappa)$.

Higher order entropy estimates (10)

- **Quantity χ**

$$\chi = \|\log T\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty} = \|\tau\|_{BMO} + \|w\|_{L^\infty}$$

- **Conditional entropicity for $\gamma^{[k]}$**

Let $k \geq 1$ and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx \leq c\, \chi \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx$$

- **Sketch of the Proof**

The quantities $T^{\sigma-\varkappa}\partial_T^\sigma\lambda$ and $T^{\sigma-\varkappa}\partial_T^\sigma\eta$ are uniformly bounded and for χ small enough the operators $T^\theta R_i R_j T^{-\theta}$ are continuous over L^2 and

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \leq c\chi^{N_\nu-1} \left(\|T^{\frac{\theta}{2}} \frac{\partial^{k+1}T}{T}\|_{L^2} + \|T^{\frac{\theta}{2}} \frac{\partial^{k+1}v}{\sqrt{T}}\|_{L^2} \right)$$

$$\|\Pi_\nu^{(k+1)}\|_{L^2}\|\Pi_\mu^{(k+1)}\|_{L^2} \leq c\chi^{N_\nu+N_\mu-2} \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k, n)$, and $N_\nu + N_\mu - 2 = \sum_{1\leq|\alpha|\leq k+1}(\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) - 2 \geq 1$

Higher order entropy estimates (11)

- **Conditional entropicity for $\tilde{\gamma}^{[k]}$**

Let $k \geq 1$ and (v,T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}|\, dx \leq c\, \chi \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx$$

- **Sketch of the Proof**

Similar proof than for $\gamma^{[k]}$ and use of weighted interpolation for the estimates

$$\big\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\big\|_{L^2} \leq c\chi^{N_\mu+N_\iota-1} \left(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\right)$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k,n)$ so that finally

$$\big\|\Pi_{\nu}^{(k+1)}\big\|_{L^2}\big\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\big\|_{L^2} \leq c\chi^{N_\nu+N_\mu+N_\iota-2} \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx$$

where $N_\nu + N_\mu + N_\iota - 2 = \sum_{1\leq|\alpha|\leq k+1}(\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) - 2 \geq 1$

Higher order entropy estimates (12)

- **Corollary**

Let $k \geq 1$ be an integer and (v,T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + (1 - c\,\chi) \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx \leq \int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[k]}\, dx + (1 - c\,\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx \leq \int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}|\, dx$$

- **Remark**

Letting $\rho\underline{b}_k = \underline{\mathfrak{a}} \min(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$ and $\rho\overline{b}_k = \overline{\mathfrak{a}} \max(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$ we have

$$2\underline{b}_k\, \gamma^{[k+1]} \leq \pi_\gamma^{[k]}\, T^{-(a_{k+1}-a_k+\varkappa)} \leq 2\overline{b}_k\, \gamma^{[k+1]}$$

$$2\underline{b}_k\, \tilde{\gamma}^{[k+1]} \leq \pi_{\tilde{\gamma}}^{[k]}\, e^{-(a_{k+1}-a_k+\varkappa)\tau} \leq 2\overline{b}_k\, \tilde{\gamma}^{[k+1]}$$

- **Parallel with heat equation**

Letting $\zeta^{[k]} = |\partial^k u|^2$ we have $\partial_t \int_{\mathbb{R}^n} \zeta^{[k]}\, dx + 2 \int_{\mathbb{R}^n} |\partial_x^2 u|^{k+1}\, dx = 0$

Higher order entropy estimates (13)

- **Estimates of convective terms**

Let $k \geq 1$ and (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$ we have

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx \leq c\chi \sup_{\mathbb{R}^n}\{T^{(1-2\varkappa+a_{k-1}-a_k)/2}\} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx\Big)^{\frac{1}{2}}$$

$$\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}|\, dx \leq c\chi \sup_{\mathbb{R}^n}\{e^{(1-2\varkappa+a_{k-1}-a_k)\tau/2}\} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx\Big)^{\frac{1}{2}}$$

- **Estimates for $\gamma^{[0]}$ and $\tilde{\gamma}^{[0]}$**

Let $0 < a_0 \leq 1$ then $\gamma^{[0]} = \tilde{\gamma}^{[0]} \geq 0$ and there exists constants $\delta_0 > 0$, $\underline{b}_0'$ and c such that for $\chi < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \underline{b}_0' \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leq 0$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + (\underline{b}_0' - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx \leq 0$$

Higher order entropy estimates (14)

- **Natural scale of temperature weights**

  Simplified coefficients $\mathrm{A}_{\lambda}^{[k]} = 1$, and $\mathrm{A}_{\eta}^{[k]} = 1/r,\ k \geq 0$

  Eliminate the $\sup T^{z}$ factors from convective terms estimates

$$a_k = a_0 + k(1 - 2\varkappa), \qquad k \geq 0$$

  Corresponds to the scale given by the Kinetic Theory since $(\eta/\rho\sqrt{rT})^{(2k)} \sim 1/T^{k(1-2\varkappa)}$

  There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{1-\varkappa}\gamma^{[k+1]}\, dx \leq c\chi \int_{\mathbb{R}^n} T^{1-\varkappa}\gamma^{[k]}\, dx$$

- **Higher order entropy estimates**

  Let (v,T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ and $\delta_{\mathrm{N}}(k,n)$ such that for $\chi < \delta_{\mathrm{N}}$ we have

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{1-\varkappa}\Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]}\Big)\, dx \leq 0$$

Higher order entropy estimates (15)

- **Uniform scale of temperature weights**

  Simplified coefficients $\mathrm{A}_{\lambda}^{[k]} = 1$, and $\mathrm{A}_{\eta}^{[k]} = 1/r,\ k \geq 0$ and simplified temperature exponents

$$a_k = a_0, \qquad k \geq 0$$

  Control of the $\sup T^z$ factors from convective terms estimates with $z < 0$ and $T \geq T_{\min} > 0$ and this requires $1 - 2\varkappa \leq 0$. There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k+1]}\, dx \leq c\chi \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k]}\, dx$$

- **Higher order entropy estimates**

  Let (v, T) be a smooth solution of the incompressible Navier-Stokes equations. There exists positive constants $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ and $\delta_{\mathrm{U}}(k,n)$ such that for $\chi < \delta_{\mathrm{N}}$ we have

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]}\Big)\, dx \leq 0$$

$$\partial_t \int_{\mathbb{R}^n} \Big(\tilde{\gamma}^{[0]} + \tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\tilde{\gamma}^{[1]} + \tilde{\gamma}^{[2]} + \cdots + \tilde{\gamma}^{[k+1]}\Big)\, dx \leq 0$$

Higher order entropy estimates (16)

- **Condition for entropicity**

  The quantity

$$\chi = \|\log T\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty} = \|\tau\|_{BMO} + \|w\|_{L^\infty}$$

  is invariant under various change of scales

  The variables τ and w can be seen as renomalized variables and also naturally arise in Maxwellian distribution functions

  We have formally $\chi = \mathcal{O}(\mathrm{Ma})$ for small Mach number flows

Application to asymptotic stability (1)

- **Incompressible model with temperature dependent transport coefficients**
- **Further assumptions**

Uniform scale of temperature weights with uniform logarithmic scaling

$$a_k = 1, \qquad k \geq 0$$

The transport coefficients satisfy the natural assumptions given by the Kinetic Theory with

$$\frac{1}{2} \leq \varkappa$$

Flow spanning the whole space and 'Constant at infinity'

We denote by v the combined unknown $\mathrm{v} = (v, T)$ and $\mathrm{v}_\infty = (0, T_\infty)$ the equilibrium point

We denote by $\mathcal{O}_\mathrm{v} = \mathbb{R}^n \times (0, \infty)$ the natural domain for the variable v, where $n \geq 2$.

Application to asymptotic stability (2)

- **Local existence**

Let $n \geq 2$ and $l \geq [n/2] + 2$ be integers and let $b > 0$ be given. Let $\mathcal{O}_0$ be an open bounded convex set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathrm{v}}$, d_1 with $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{v}})$, and define $\mathcal{O}_1 = \{ \, \mathrm{v} \in \mathcal{O}_{\mathrm{v}}; \; d(\mathrm{v}, \overline{\mathcal{O}}_0) < d_1 \, \}$.

There exists $\bar{t} > 0$ small enough, which only depend on $\mathcal{O}_0$, d_1, and b, such that for any v_0 with $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l} < b$ and $\mathrm{v}_0 \in \overline{\mathcal{O}}_0$, there exists a unique local solution $\mathrm{v} = (v, T)$ to the equation system with initial condition

$$\mathrm{v}(0, x) = \mathrm{v}_0(x)$$

such that

$$\mathrm{v}(t, x) \in \mathcal{O}_1$$

and

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0, \bar{t}\,], H^l(\mathbb{R}^n)\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}(\mathbb{R}^n)\big) \cap L^2\big((0, \bar{t}\,), H^{l+1}(\mathbb{R}^n)\big)$$

In addition, denoting for short $\mathrm{v}(t) = \mathrm{v}(t, \cdot)$, there exists $C(\mathcal{O}_0, d_1, b) > 0$ such that

$$\sup_{0 \leq s \leq \bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^{l+1}}^2 \, ds \leq C \|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l}^2$$

Application to asymptotic stability (3)

- **Sketch of the Proof**

Solutions are fixed points $\widetilde{\mathrm{v}} = \mathrm{v}$ of the linear system of equations in $\widetilde{\mathrm{v}} = (\widetilde{v}, \widetilde{T})$

$$\partial_t(\rho\widetilde{v}) - \mathbb{P}\big(\partial_x\cdot\big(\eta(T)\,\partial_x\widetilde{v}\big)\big) = \mathbb{P}\big(f_v(\mathrm{v}, \partial_x\mathrm{v})\big)$$

$$\partial_t(\rho c_v\widetilde{T}) - \partial_x\cdot\big(\lambda(T)\,\partial_x\widetilde{T}\big) = f_T(\mathrm{v}, \partial_x\mathrm{v})$$

with $f_v = -\partial_x\cdot(\rho v\otimes v) + \partial_T\eta(T)\partial_x T\cdot\partial_x v^t$ and $f_T = -\partial_x\cdot(\rho c_v T v) + \frac{1}{2}\eta(T) d{:}d$. We have the estimates

$$\|\widetilde{\mathrm{v}}(t) - \mathrm{v}_\infty\|^2_{H^k} + \int_0^t \|\widetilde{\mathrm{v}}(s) - \mathrm{v}_\infty\|^2_{H^{k+1}}\,ds \le C_1^2 \exp\big(C_2(t + M_1\sqrt{t}\,)\big)\big(\|\mathrm{v}_0 - \mathrm{v}_\infty\|^2_{H^k} + C_2\int_0^t \|f(s)\|^2_{H^{k-1}}\,ds\big)$$

where $C_1 = C_1(\mathcal{O}_1)$, $C_2 = C_2(\mathcal{O}_1, M)$ and

$$\sup_{0\le s\le \bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|^2_{H^l} + \int_0^{\bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|^2_{H^{l+1}}\,ds \le M^2 \qquad \int_0^{\bar{t}} \|\partial_t\mathrm{v}(s)\|^2_{H^{l-1}}\,ds \le M_1^2$$

These a priori estimates for linear equations are obtained by deriving the governing equations, multiplying by the derivative of the solution, and using the properties of the Leray projector $\mathbb{P}$. Existence of such solutions $\widetilde{\mathrm{v}}$ to the linear equations are obtained from a priori estimates by Galerkin approximations.

Application to asymptotic stability (4)

- **Regularity of the local solution**

Assume that for some $k \geq l$ we have $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$. Then

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0,\bar{t}\,], H^k(\mathbb{R}^n)\big) \cap C^1\big([0,\bar{t}\,], H^{k-2}(\mathbb{R}^n)\big) \cap L^2\big((0,\bar{t}\,), H^{k+1}(\mathbb{R}^n)\big)$$

In particular, v is smooth when $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

- **Diffeomorphism $\mathcal{F}$**

Denote by $\mathcal{F} : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^{n+1}$ the application defined by $\mathcal{F}(\mathrm{v}) = \mathrm{w} = (w,\tau)$, that is, $\mathcal{F}(v,T) = (w,\tau) = (v/\sqrt{T}, \log T)$. Then $\mathcal{F}$ is a C^∞ diffeomorphism and its jacobian matrix reads

$$\partial_{\mathrm{v}}\mathcal{F} = \begin{pmatrix} \frac{\mathbb{I}}{\sqrt{T}} & -\frac{1}{2}\frac{v}{T^{3/2}} \\ 0 & \frac{1}{T} \end{pmatrix}$$

In addition, for any $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, defining $\widetilde{\mathcal{O}} = (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$, the corresponding open set $\mathcal{O} = \mathcal{F}^{-1}(\widetilde{\mathcal{O}})$ is convex.

Application to asymptotic stability (5)

- **Transport of the local solution**

Let $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, $\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$, $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$, let $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{V}})$, denote $\mathcal{O}_1 = \{\, \mathrm{V} \in \mathcal{O}_{\mathrm{V}};\ d(\mathrm{V}, \overline{\mathcal{O}}_0) < d_1 \,\}$, and select an arbitrary $b > 0$. The local solution built with $\mathcal{O}_0$, d_1, and b is such that

$$\mathrm{W} - \mathrm{W}_\infty \in C^0\big([0,\bar{t}\,], H^l(\mathbb{R}^n)\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}(\mathbb{R}^n)\big) \cap L^2\big((0,\bar{t}\,), H^{l+1}(\mathbb{R}^n)\big)$$

and there exists $C > 0$ which only depend on $\mathcal{O}_0$, d_1, and b, such that

$$\sup_{0 \le s \le \bar{t}} \|\mathrm{W}(s) - \mathrm{W}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathrm{W}(s) - \mathrm{W}_\infty\|_{H^{l+1}}^2 \, ds \le C \|\mathrm{W}_0 - \mathrm{W}_\infty\|_{H^l}^2$$

Moreover, the kinetic entropy estimators are such that $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0,\bar{t}\,], L^1(\mathbb{R}^n)\big)$.

- **Estimates for** W

There exists a constant $\underline{c}_\Gamma(T_{\min})$ such that for any $k \ge 0$ and any W with $\mathrm{W} - \mathrm{W}_\infty \in H^k$ we have

$$\underline{c}_\Gamma \|\mathrm{W} - \mathrm{W}_\infty\|_{H^k}^2 \le \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]} \, dx$$

Application to asymptotic stability (6)

- **Global solutions**

Let $n \geq 2$ and $l \geq [n/2] + 2$ be integers. Assume that the coefficients λ and η are suth that $\varkappa \geq 1/2$. There exists $\delta_\Gamma(l, n, T_{\min}) > 0$ such that for T_0 and v_0 satisfying $T_{\min} \leq \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$, $k \in \mathbb{N}$, and

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]} \, dx \leq \delta_\Gamma$$

there exists a unique global solution $\mathrm{v} = (v, T)$ such that

$$\mathrm{v} - \mathrm{v}_\infty, \mathrm{w} - \mathrm{w}_\infty \in C\big([0, \infty), H^l(\mathbb{R}^n)\big) \cap C^1\big([0, \infty), H^{l-2}(\mathbb{R}^n)\big)$$
$$\partial_x \mathrm{v}, \partial_x \mathrm{w} \in L^2\big((0, \infty), H^l(\mathbb{R}^n)\big)$$

and we have the estimates

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]} \, dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} T^\varkappa \big(\widetilde{\Gamma}^{[l+1]} - \tilde{\gamma}^{[0]}\big) \, dx dt \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]} \, dx$$

where $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$. Furthermore, this solution is smooth and we have

$$\lim_{t \to \infty} \|\mathrm{v}(t, \cdot) - \mathrm{v}_\infty\|_{L^\infty} = 0$$

Application to asymptotic stability (7)

- **Sketch of the Proof**

We have the inequalities

$$\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} \leq \|\tau - \tau_\infty\|_{L^\infty} + \|w\|_{L^\infty} \leq c_0 \|\mathrm{w} - \mathrm{w}_\infty\|_{H^{l_0}} \leq \frac{c_0}{\sqrt{\underline{c}_\Gamma}} \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx$$

We set

$$\delta_\Gamma = \frac{\delta_\mathrm{U}^2}{4c_0^2} \underline{c}_\Gamma$$

so that

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx < \delta_\Gamma \Longrightarrow \chi < \delta_\mathrm{U}/2$$

Higher order entropy estimates yields that

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx$$

Local existence used repeatedly

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Conclusion and future work (1)

- **Incompressible models**

  Asymptotic analysis for small Mach numbers

- **Compressible models**

  Higher order entropy estimates

  Application to asymptotic stability

  Asymptotic analysis for small Mach numbers

- **Zero Mach number models**

  Higher order entropy estimates

  Application to asymptotic stability

- **Multicomponent flows**