

# A Non-Local Transport Equation modelling Dislocations Dynamics

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- 1 Presentation of the Physical Model
- 2 First Main Result
- 3 Main Ideas for Proof of the First Result
- 4 Error Estimate
- 5 Numerical Simulations
- 6 Conclusion and Perspectives

# Presentation of the Physical Model

Dislocations are line defects in a crystal material (Hirth & Lothe, 1992).

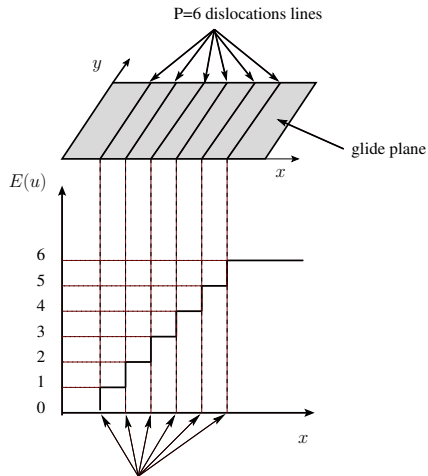


Figure: Representation of dislocations with the function  $E(u)$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(x, t) = c[u](x, t) \frac{\partial u}{\partial x}(x, t) \quad \text{in } \mathbb{R} \times (0, +\infty) \\ c[u](x, t) = c^{\text{ext}}(x) + \int_{\mathbb{R}} c^0(x') E((u(x - x', t))) \, dx' \\ u(x, 0) = u^0(x) \quad \text{in } \mathbb{R} \end{array} \right. \quad (1)$$

where  $E$  is the floor function.

Dislocations move with a non-local velocity  $c[u]$ . It is the sum of two terms:

- $c^{\text{ext}}$  represents the exterior stress created by obstacles (such as precipitates, other defects, ...).
- The second term is non-local and represents the elastic interior stress created by all the dislocations in the material.

We make the following assumptions for the exterior stress  $c^{\text{ext}}$  and the kernel  $c^0$ :

$$\begin{cases} c^{\text{ext}} \in W^{1,\infty}(\mathbb{R}) \text{ such that } c^{\text{ext}}(x+1) = c^{\text{ext}}(x) \text{ in } \mathbb{R}, \\ c^0 \in C_0^\infty(\mathbb{R}) \text{ such that } c^0(x) = c^0(-x) \text{ and } \int_{\mathbb{R}} c^0(x) dx = 0. \end{cases} \quad (2)$$

We consider the initial condition  $u^0 \in \text{Lip}(\mathbb{R})$  such that for  $x \in \mathbb{R}$

$$u^0(x+1) = u^0(x) + P \quad \text{and} \quad 0 < b_0 \leq \frac{\partial u^0}{\partial x} \leq B_0 < +\infty \quad \text{a.e.} \quad (3)$$

with  $b_0$  and  $B_0$  some constants and  $P \in \mathbb{N} \setminus \{0\}$ . A natural case to study the solutions of (1) is the continuous viscosity solutions (Barles, 1994).

## Theorem 1

Under Assumptions (2) and (3), there exists a unique continuous viscosity solution  $u \in W_{\text{loc}}^{1,\infty}(\mathbb{R} \times (0, +\infty))$  of (1) such that

$$u(x+1, t) = u(x, t) + P \quad \forall (x, t) \in \mathbb{R} \times (0, +\infty). \quad (4)$$

We prove the previous theorem in two steps.

- 1 First, we prove the result for a short time (Alvarez and *al.*, 2005; Ghorbel and *al.*, 2005) using a fixed point theorem.

Given  $T > 0$  and a function  $v$  satisfying (4), we consider the function  $w = \phi(v)$  satisfying (4), solution of

$$\frac{\partial w}{\partial t}(x, t) = c[v](x, t) \frac{\partial w}{\partial x}(x, t) \quad \text{in } \mathbb{R} \times (0, +\infty). \quad (5)$$

For  $T$  chosen sufficiently small, we prove that the map  $\phi$  is a contraction in a well chosen space.

- 2 Secondly, we repeat this short time result on a sequence of time intervals of lengths  $T_n$  decreasing to zero, such that  $\sum_{n \in \mathbb{N}} T_n = +\infty$ .

Given a mesh size  $\Delta x$ ,  $\Delta t$  and a lattice  $l_d = \{(i\Delta x, n\Delta t); i \in \mathbb{Z}, n \in \mathbb{N}\}$ ,  $(x_i, t_n)$  denotes the node  $(i\Delta x, n\Delta t)$  and  $v^n = (v_i^n)_i$  the values of the numerical approximation of the continuous solution  $u(x_i, t_n)$ .

$$v_i^0 = u^0(x_i), \quad v_i^{n+1} = v_i^n + \Delta t c_i(v^n) \times \begin{cases} \frac{v_{i+1}^n - v_i^n}{\Delta x} & \text{if } c_i(v^n) \geq 0 \\ \frac{v_i^n - v_{i-1}^n}{\Delta x} & \text{if } c_i(v^n) < 0 \end{cases} \quad (6)$$

We choose  $\Delta x = \frac{1}{K}$ ,  $K \in \mathbb{N} \setminus \{0\}$  because of the 1-periodicity in space. We denote  $c_i^{\text{ext}} = c^{\text{ext}}(x_i)$  which satisfies  $c_{i+K}^{\text{ext}} = c_i^{\text{ext}}$ . The discrete velocity is

$$c_i(v^n) = c_i^{\text{ext}} + \sum_{l \in \mathbb{Z}} c_l^0 E(v_{i-l}^n) \Delta x \quad (7)$$

where

$$c_i^0 = \frac{1}{\Delta x} \int_{l_i} c^0(x) dx \quad \text{and} \quad l_i = \left[ x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2} \right]. \quad (8)$$

We assume the CFL (Courant, Friedrichs, Levy) condition. Then

## Theorem 2

there exists two constants  $T, C > 0$  such that

$$\sup_{i \in \mathbb{Z}} |u(i\Delta x, n\Delta t) - v_i^n| \leq C\sqrt{\Delta x} \quad \text{for all } n \leq \frac{T}{\Delta t}.$$

## Main Idea of Proof of Theorem 2

We apply the technical reasoning of Alvarez and *al.*, 2005.



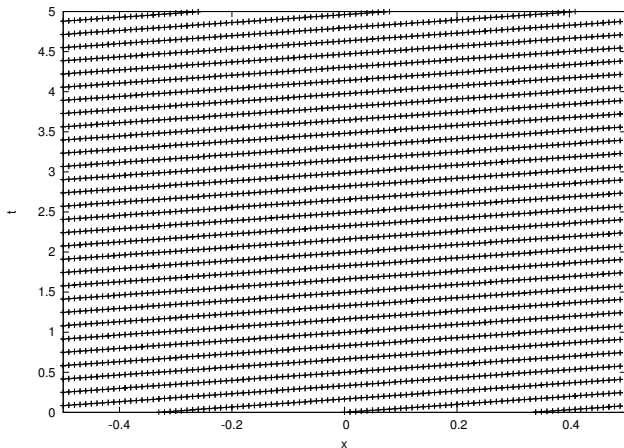


Figure: Linear:  $\Delta x = 0.01$ ,  $\Delta t = 2.438 \cdot 10^{-3}$

# Dynamics of several dislocations through obstacles

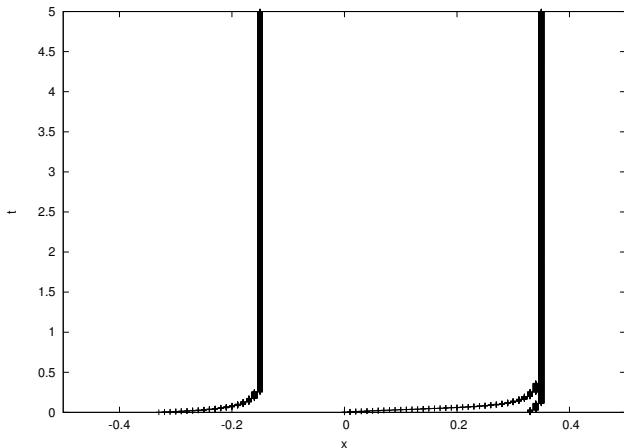


Figure: Trapping:  $\Delta x = 0.01$ ,  $\Delta t = 1.239 \cdot 10^{-3}$

# Dynamics of several dislocations through obstacles

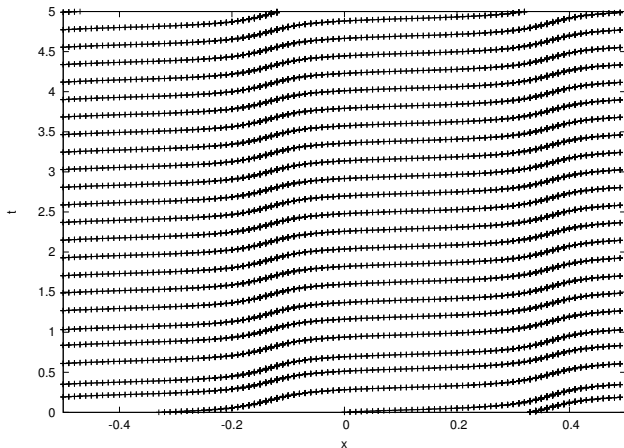


Figure: Pile-Up:  $\Delta x = 0.01$ ,  $\Delta t = 1.102 \cdot 10^{-3}$

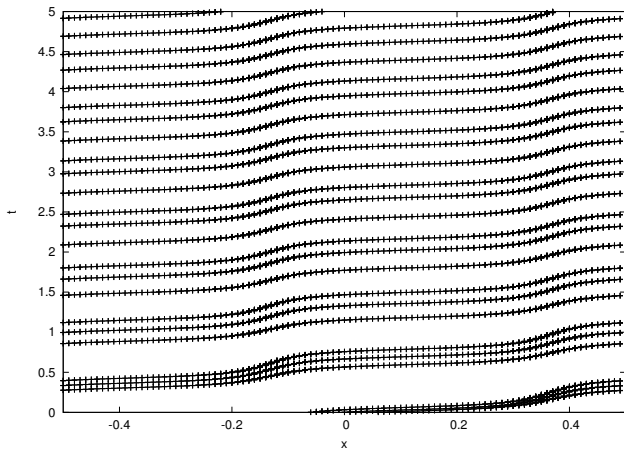


Figure: Pile-Up:  $\Delta x = 0.01$ ,  $\Delta t = 1.10219 \cdot 10^{-3}$

- Homogenization of a large number of dislocations through obstacles.
- Friction of dislocations.
- Homogenization of walls of dislocations.  
Joint works with P. Hoch (CEA) and R. Monneau (Cermics).