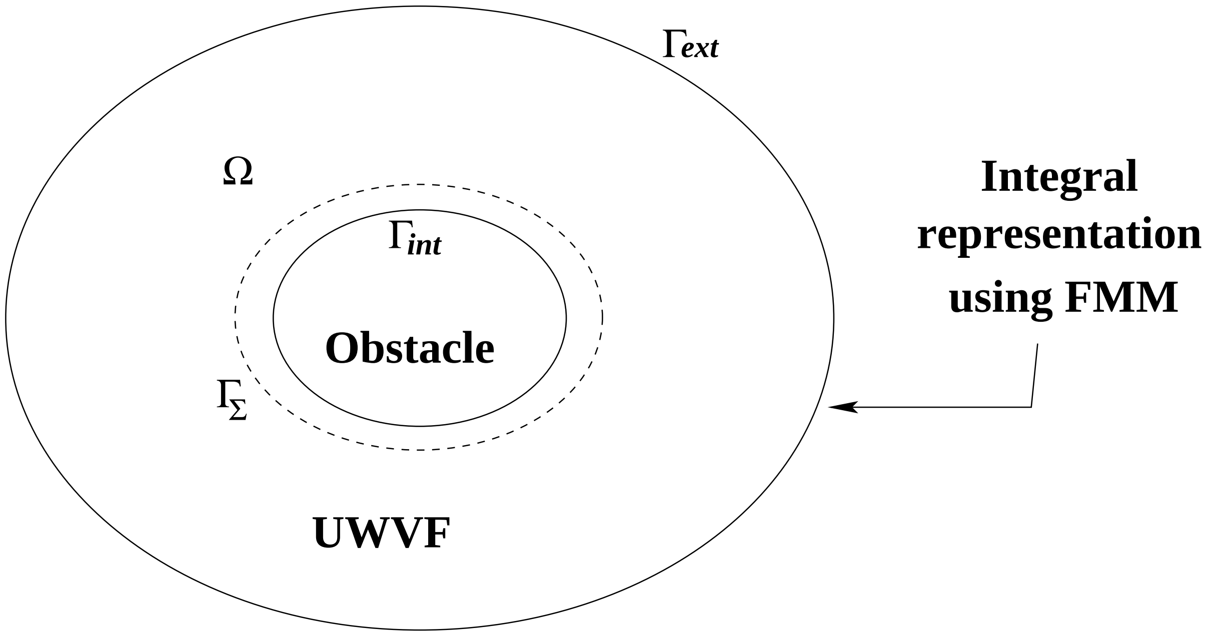
Ultra-Weak Variational Formulation and Integral Representation using a Fast Multipole Method for the Equations of Electromagnetism

Eric Darrigrand and Peter Monk

Université de Rennes 1 **University of Delaware**



Ω: “exterior” domain

Γ_Σ or Γ_{int}: Boundary of an obstacle

Γ_{ext}: Artificial boundary

Time-Harmonic Maxwell's Equations

For given m, j such that $\nabla \cdot m = 0 = \nabla \cdot j$, find (E, H) such that, in Ω

$$\begin{cases} \nabla \wedge E - \imath\omega\mu H = -m\,, \\ \nabla \wedge H + \imath\omega\varepsilon E = j\,, \\ \nabla \cdot (\varepsilon E) = 0\,, \\ \nabla \cdot (\mu H) = 0\,, \end{cases}$$

with the Boundary Condition on $\Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$

$$-\mid \sqrt{\varepsilon} \mid E \wedge \nu + (\mid \sqrt{\mu} \mid H \wedge \nu) \wedge \nu = Q(\mid \sqrt{\varepsilon} \mid E \wedge \nu + (\mid \sqrt{\mu} \mid H \wedge \nu) \wedge \nu) + g \quad ,$$

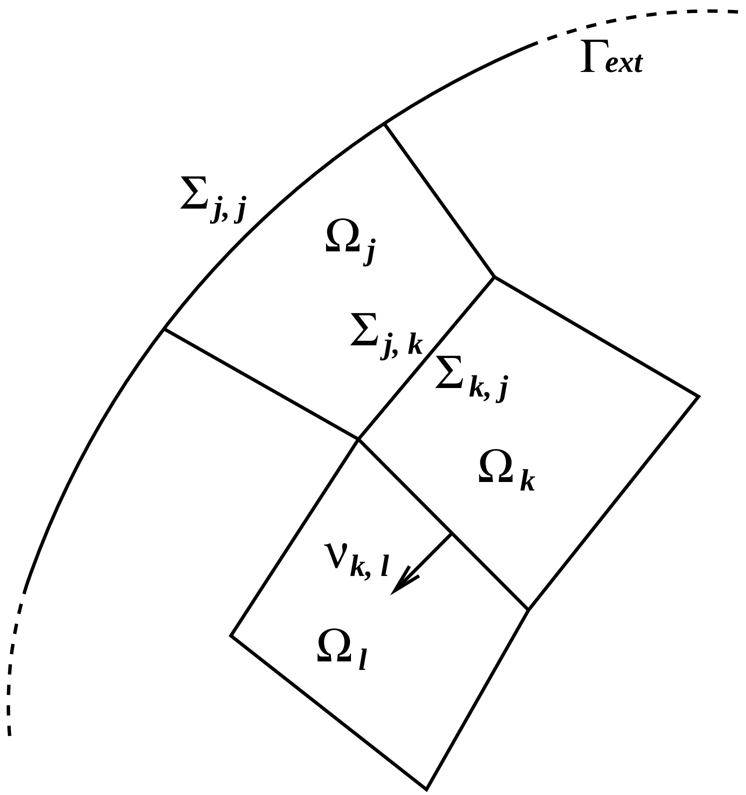
where

$Q = 0$ on Γ_{ext} (ABC) and g involving the incident wave.

Ultra-Weak Variational Formulation (UWVF)

O. Cessenat - B. Després

Decomposition of the domain: K sub-domains $\{\Omega_k\}_{k=1,\dots,K}$.



$$\partial\Omega_k = \bigcup_j \Sigma_{k,j}$$

$\nu_k =$ exterior normal to Ω_k

$\nu_{kj} =$ normal to $\Sigma_{k,j}$ from Ω_k to Ω_j

$\varepsilon_k = \varepsilon_{/\Omega_k}$

$\varepsilon_{kj} = |\sqrt{\varepsilon_k \varepsilon_j}|$

$\varepsilon_{kk} = |\varepsilon_k|$

Variational Space: $V = \prod_{k=1}^{K} L_t^2(\partial\Omega_k)$.

Usual Scalar Product: $(\mathcal{X}, \mathcal{Y})_V = \sum_k \int_{\partial\Omega_k} \mathcal{X}_{/\partial\Omega_k} \overline{\mathcal{Y}_{/\partial\Omega_k}}$.

Assumption: ε and μ constant on each Ω_k, (E, H) found such that:

$$\begin{aligned} E_k, H_k \in \; & \widetilde{\mathcal{H}}(\Omega_k, \partial\Omega_k) \\ = \; & \{F \in (H(\mathrm{rot}, \Omega_k) \cap H(\mathrm{div}_0, \Omega_k)); F_{/\partial\Omega_k} \wedge \nu_k \in L_t^2(\partial\Omega_k)\} \end{aligned}$$

The unknown of the UWVF: $\mathcal{X} \in V$ defined by $\mathcal{X}_{/\partial\Omega_k} = \mathcal{X}_k \in L_t^2(\partial\Omega_k)$ and

$$(\mathcal{X}_k)_{/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}(E_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H_k \wedge \nu_k) \wedge \nu_k)$$

Variational Formulation: Find $\mathcal{X} \in V$ such that

$$(\mathcal{X}, \mathcal{Y})_V - (\Pi\mathcal{X}, F\mathcal{Y})_V = (b, \mathcal{Y})_V$$

for all **test function** $\mathcal{Y} \in V$ defined by

$$(\mathcal{Y}_k)_{/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k)$$

such that $(E'_k, H'_k) \in (\widetilde{\mathcal{H}}(\Omega_k, \partial\Omega_k))^2$ and

$$\begin{cases} \nabla \wedge E'_k - \imath\omega\overline{\mu_k}H'_k = 0\,, \qquad \nabla \wedge H'_k + \imath\omega\overline{\varepsilon_k}E'_k = 0\,, \\ \sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k) \in L^2_t(\partial\Omega_k)\,, \end{cases}$$

and with the **operators** $F : V \to V$ and $\Pi : V \to V$ defined by

$$(F\mathcal{Y})_{/\Sigma_{k,j}} = -\sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k)$$

$$(\Pi\mathcal{Y})_{/\Sigma_{k,j}} = \mathcal{Y}_{/\Sigma_{j,k}} \qquad , \qquad (\Pi\mathcal{Y})_{/\Sigma_{k,k}} = Q\mathcal{Y}_{/\Sigma_{k,k}}$$

Galerkin Discretisation

Find $\mathcal{X}_h \in V_h$ such that $\forall \mathcal{Y}_h \in V_h$

$$(\mathcal{X}_h, \mathcal{Y}_h)_{V_h} - (\Pi \mathcal{X}_h, F \mathcal{Y}_h)_{V_h} = (b, \mathcal{Y}_h)_{V_h}$$

Basis Functions $Z_i, i \in J \quad \rightarrow \quad \mathcal{X}_h = \sum_{i \in J} X_i Z_i.$

Find $(X_i)_{i \in J} \in \mathbb{C}^{\# J}$ such that

$$(D - C)X = b$$

with

$$D_{kj} = \delta_{kj}(Z_j, Z_k)_{V_h} \qquad \text{and} \qquad C_{kj} = (\Pi Z_j, F Z_k)_{V_h}$$

Boundary condition included in C $(k = j)$.

Choice of Z_i: On each element Ω_k

$$\text{for } l = 1, ..., p \, , \quad Z_{kl/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}\, E^F_{kl} \wedge \nu_k + \sqrt{\mu_{kj}}\, (H^F_{kl} \wedge \nu_k) \wedge \nu_k \quad ,$$

$$\text{for } l = p+1, ..., 2p \, , \quad Z_{kl/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}\, E^G_{kl} \wedge \nu_k + \sqrt{\mu_{kj}}\, (H^G_{kl} \wedge \nu_k) \wedge \nu_k \quad ,$$

with

$$E^F_{kl}(X) = \sqrt{\overline{\mu_{kj}}}\, F_{kl}\, e^{\imath\, \omega \sqrt{\overline{\varepsilon_k}\, \overline{\mu_k}}\, (V_{kl} \cdot X)} \, , \quad H^F_{kl}(X) = \imath \sqrt{\overline{\varepsilon_{kj}}}\, F_{kl}\, e^{\imath\, \omega \sqrt{\overline{\varepsilon_k}\, \overline{\mu_k}}\, (V_{kl} \cdot X)} \, ,$$
$$E^G_{kl}(X) = \sqrt{\overline{\mu_{kj}}}\, G_{kl}\, e^{\imath\, \omega \sqrt{\overline{\varepsilon_k}\, \overline{\mu_k}}\, (V_{kl} \cdot X)} \, , \quad H^G_{kl}(X) = -\imath \sqrt{\overline{\varepsilon_{kj}}}\, G_{kl}\, e^{\imath\, \omega \sqrt{\overline{\varepsilon_k}\, \overline{\mu_k}}\, (V_{kl} \cdot X)} \, .$$

and

$$F_{kl} = E^0_{kl} + \imath\, E^0_{kl} \wedge V_{kl} \quad , \quad G_{kl} = E^0_{kl} - \imath\, E^0_{kl} \wedge V_{kl} \quad .$$

where $E^0_{kl} \perp V_{kl}$

Integral Representation

C. Hazard - M. Lenoir ; J. Liu - J.-M. Jin

Previous ABC on the artificial boundary

$$-E \wedge \nu + (H \wedge \nu) \wedge \nu = -E_0 \wedge \nu + (H_0 \wedge \nu) \wedge \nu$$

Ideal Boundary Condition on $\Gamma_{\rm ext}$

$$-E \wedge \nu + (H \wedge \nu) \wedge \nu = -E^s \wedge \nu + (H^s \wedge \nu) \wedge \nu - E_0 \wedge \nu + (H_0 \wedge \nu) \wedge \nu$$

On $\Gamma_{\rm ext}$, (E^s, H^s) given by **the Stratton-Chu formulae**

$$E^s(x) = \nabla_x \wedge \int_{\Gamma_{\rm int}} G(x,y)\, \nu_{\rm int}(y) \wedge E(y)\, d\gamma(y)$$

$$- \frac{1}{\imath\omega} \nabla_x \wedge \nabla_x \wedge \int_{\Gamma_{\rm int}} G(x,y)\, \nu_{\rm int}(y) \wedge H(y)\, d\gamma(y) \quad ,$$

$$H^s(x) = \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x,y)\,\nu_{\text{int}}(y) \wedge H(y)\,d\gamma(y)$$

$$+ \quad \frac{1}{\imath\omega} \nabla_x \wedge \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x,y)\,\nu_{\text{int}}(y) \wedge E(y)\,d\gamma(y) \quad ,$$

On Γ_{int}

$$\nu \wedge E = \frac{Q-1}{2\sqrt{\varepsilon_{kk}}}\,\mathcal{X} \quad \text{and} \quad \nu \wedge H = \frac{Q+1}{2\sqrt{\mu_{kk}}}\,\mathcal{X} \wedge \nu$$

That is

$$\nu_{\text{int}} \wedge E = -\frac{Q-1}{2\sqrt{\varepsilon_{kk}}}\,\mathcal{X} \quad \text{and} \quad \nu_{\text{int}} \wedge H = -\frac{Q+1}{2\sqrt{\mu_{kk}}}\,\mathcal{X} \wedge \nu$$

The new discret system

$$(D - C - \widetilde{C})X = b \quad , \quad \widetilde{C} = \widetilde{C}_1 + \widetilde{C}_2 + \widetilde{C}_3 + \widetilde{C}_4 \quad ,$$

with, for $i = 1, ..., 4$, $\forall k, l$

$$(\widetilde{C}_i\, \mathcal{X}_h)_{kl} = \int_{\Sigma_{kk}^{\text{ext}}} \frac{\frac{1}{\sqrt{\varepsilon_{kk}\,\mu_{kk}}} S_i(\mathcal{X}_h) \cdot}{\left(-\sqrt{\varepsilon_{kk}}\, E'_{kl} \wedge \nu_k + \sqrt{\mu_{kk}}\, (H'_{kl} \wedge \nu_k) \wedge \nu_k\right) d\gamma_{\text{ext}}} \quad .$$

where

$$(S_2(\mathcal{X}))(x) = \frac{-1}{\imath\omega} \left(\omega^2 \int_{\Gamma_{\text{int}}} \frac{Q(y)+1}{2\sqrt{\mu(y)}} G(x,y)\,(\mathcal{X}(y) \wedge \nu(y))\, d\gamma(y) \right.$$

$$\left. - \int_{\Gamma_{\text{int}}} \frac{Q(y)+1}{2\sqrt{\mu(y)}} \nabla_x \nabla_y^t G(x,y)(\mathcal{X}(y) \wedge \nu(y))\, d\gamma(y) \right) \wedge \nu(x) \ ,$$

$$(S_1(\mathcal{X}))(x) = \left(- \int_{\Gamma_{\text{int}}} \frac{Q(y) - 1}{2\sqrt{\varepsilon(y)}} \nabla_y G(x,y) \wedge \mathcal{X}(y)\, d\gamma(y) \right) \wedge \nu(x) \quad ,$$

For instance

$$(\widetilde{C}_1)_{kj}^{lm} = \int_{\Sigma_{kk}^{\text{ext}}} \frac{1}{\sqrt{\varepsilon_{kk}\,\mu_{kk}}} \left(- \int_{\Sigma_{jj}^{\text{int}}} \frac{Q_j - 1}{2\sqrt{\varepsilon_{jj}}} \nabla_y G(x,y) \wedge Z_{jm}(y)\, d\gamma(y) \right) \wedge \nu(x) \cdot \mathcal{Z}_{kl}(x)\, d\gamma(x) \; .$$

with

$$Z_{jm}(y) = \sqrt{\varepsilon_{jj}}\, E_{jm}^{F/G} \wedge \nu_j + \sqrt{\mu_{jj}}\, (H_{jm}^{F/G} \wedge \nu_j) \wedge \nu_j$$

$$\mathcal{Z}_{kl}(x) = \overline{\left(-\sqrt{\varepsilon_{kk}}\, E_{kl}^{F/G} \wedge \nu_k + \sqrt{\mu_{kk}}\, (H_{kl}^{F/G} \wedge \nu_k) \wedge \nu_k \right)}$$

Numerical solution of $(D - C - \widetilde{C})X = b$

Two ways:

- In the same way as previously with $C \longleftarrow C + \widetilde{C}$

- Another way: Iterative relaxed Jacobi method

$$X^{(n)} \longleftarrow \alpha \widetilde{X}^{(n)} + (1-\alpha) X^{(n-1)}$$

  with $\widetilde{X}^{(n)}$ given by

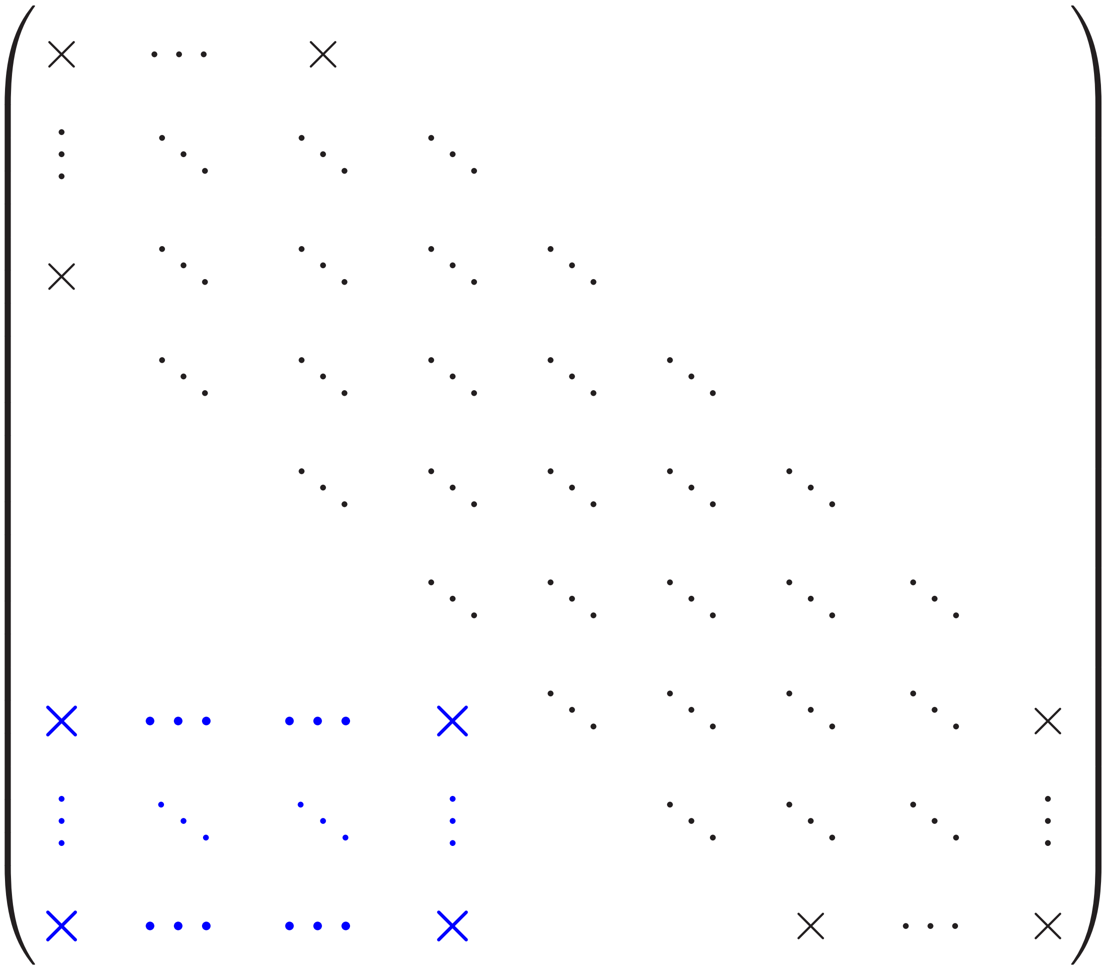
$$(D - C)\widetilde{X}^{(n)} = b + \widetilde{C} X^{(n-1)}$$

The matrix C

$$\begin{pmatrix}
\times & \cdots & \times & & & & & & \\
\vdots & \ddots & \ddots & \ddots & & & & & \\
\times & \ddots & \ddots & \ddots & \ddots & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \times \\
 & & & & & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & \times & \cdots & \times
\end{pmatrix}$$

- Black block: C
  sparse
  of length κ^3 or κ^2

The matrix $C + \widetilde{C}$



- Black block: C

  sparse

  of length κ^3 or κ^2

- Blue block: $\widetilde{C}$

  dense

  of size $\kappa^2 \times \kappa^2$

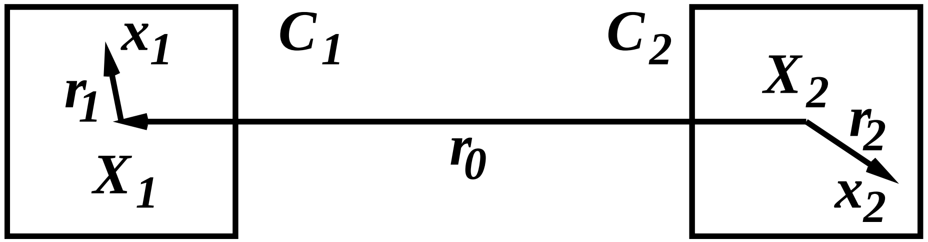
Algorithmic Complexity (1/2)

$p = \#$ of directions for the basis functions and $\kappa =$ wavenumber

Method	Number of elements	Cost of the solution
UWVF	κ^3	$\kappa^3 p^2$
UWVF+IE	κ^2	$\kappa^2 p^2 + \kappa^4 p^2$

Use of a FMM to reduce the cost related to the integral representation

FMM expansion



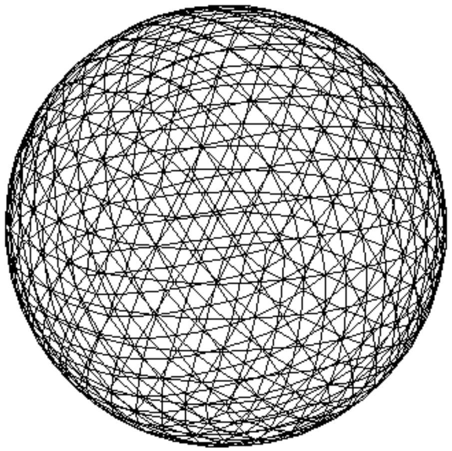
$$r_0 = X_1 - X_2 \qquad r_i = x_i - X_i$$

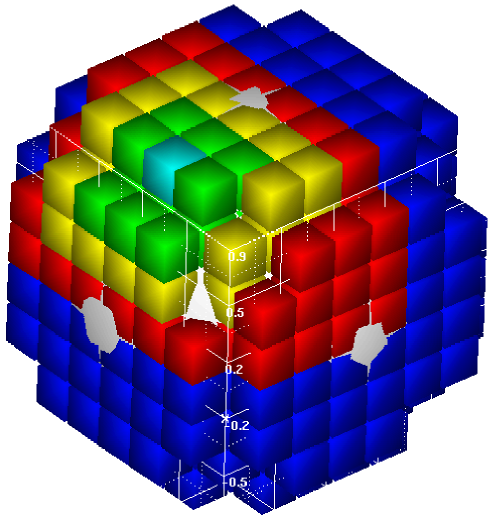
$$r = r_1 - r_2 \qquad |\boldsymbol{r}| < |\boldsymbol{r_0}|$$

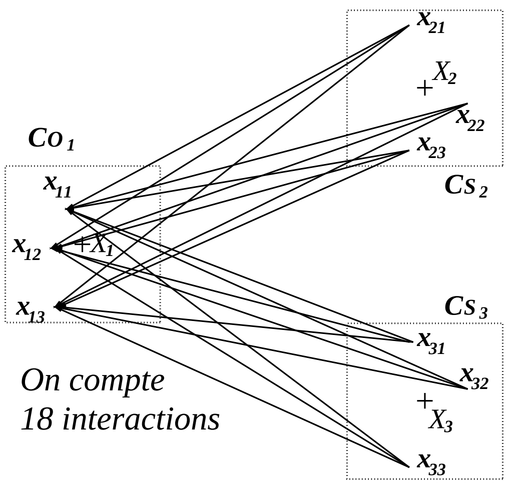
$$G(x_1, x_2) = \frac{e^{\imath\kappa|x_1 - x_2|}}{4\pi|x_1 - x_2|} \approx \frac{1}{4\pi} \sum_{p=1}^{S} \omega_p \, \boldsymbol{e^{\imath\kappa s_p \cdot r_1}} \, \mathcal{T}_{L,r_0}(s_p) \, \boldsymbol{e^{-\imath\kappa s_p \cdot r_2}}$$

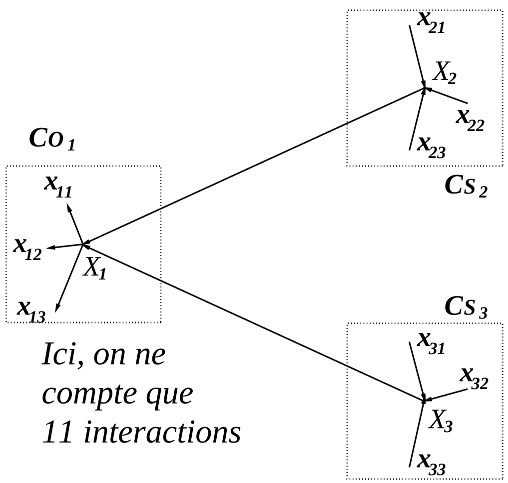
$$\mathcal{T}_{L,r_0}(s) = \imath\kappa \sum_{l=0}^{L} \frac{(2l+1)\imath^l}{4\pi} h_l^{(1)}(\kappa|r_0|) \, P_l(\cos(s, r_0))$$

- $d =$ diameter of the boxes
- $L \sim \kappa d$ (Truncation parameter of the Gegenbauer Series)
- $S \sim L^2$ (Discretization of the unit sphere in the Funck-Hecke Formula)




0.9
0.5
0.2
0.2
0.5


Co 1
x11
x12
x13
+X1
x21
+X2
x22
x23
Cs 2
Cs 3
x31
x32
+X3
x33
On compte
18 interactions


Co 1
x11
x12
x13
X1
x21
X2
x22
x23
Cs 2
Cs 3
x31
x32
X3
x33
Ici, on ne
compte que
11 interactions

Algorithmic Complexity (2/2)

$p = \#$ of directions for the basis functions and $\kappa =$ wavenumber

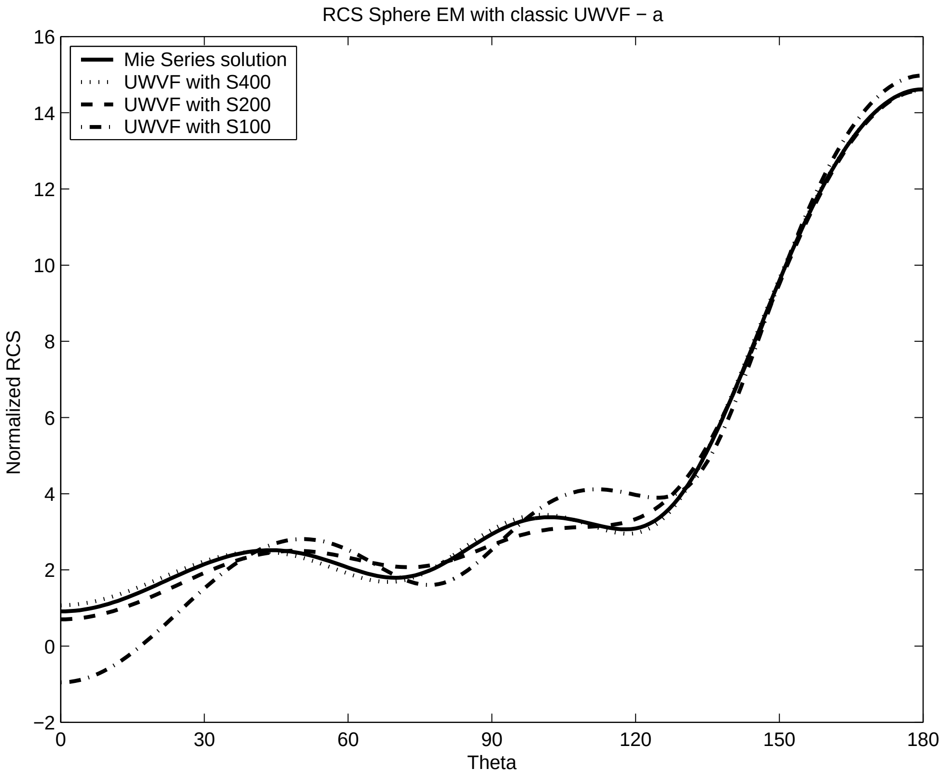
Method	Number of elements	Cost of the solution
UWVF	κ^3	$\kappa^3 p^2$
UWVF+IE	κ^2	$\kappa^2 p^2 + \kappa^4 p^2$
UWVF+IE+1LFMM	κ^2	$\kappa^2 p^2 + \kappa^3 p$
UWVF+IE+MLFMM	κ^2	$\kappa^2 p^2 + \kappa^2 \ln(\kappa) p$

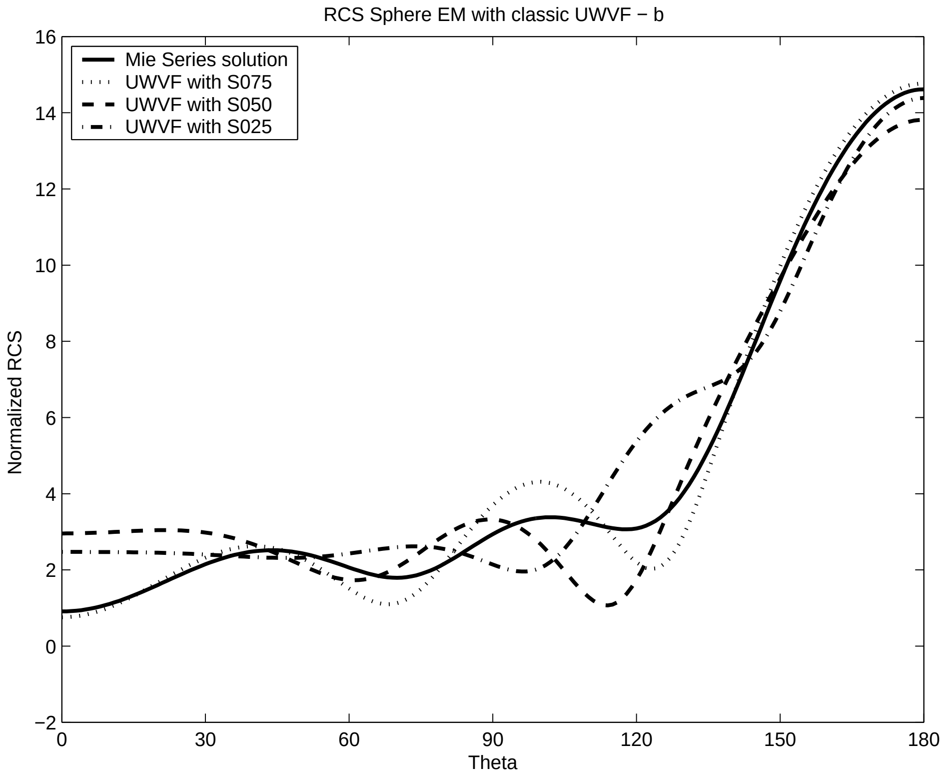
Case of the unit sphere ; $\kappa = 4$; wavelength $= \lambda = \frac{\pi}{2}$

Artificial boundary at different distances (4 ; 2 ; 1 ; 0.75 ; 0.5 ; 0.25 m).

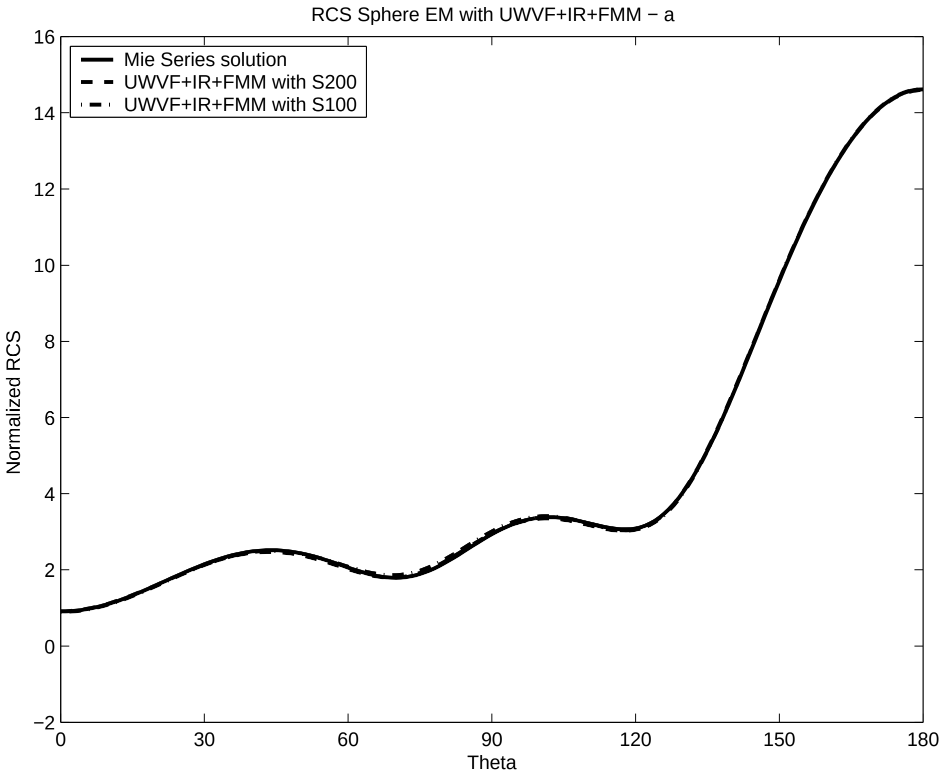
Name	S400	S200	S100	S075	S050	S025
Radius in m	5	3	2	1.75	1.5	1.25
Dist. between Γ_{int} and Γ_{ext}	$\approx 2.6\lambda$	$\approx 1.3\lambda$	$\approx 2\lambda/3$	$\approx \lambda/2$	$\approx \lambda/3$	$\approx \lambda/6$
Num. of tetrahedra	16179	14526	40609	30133	21083	11008
Num. of basis fcts per tetrahedron	8 to 128	8 to 72	10 to 32	10 to 30	8 to 28	10 to 24
Num. of DoF	880200	508450	753616	536874	356666	178146

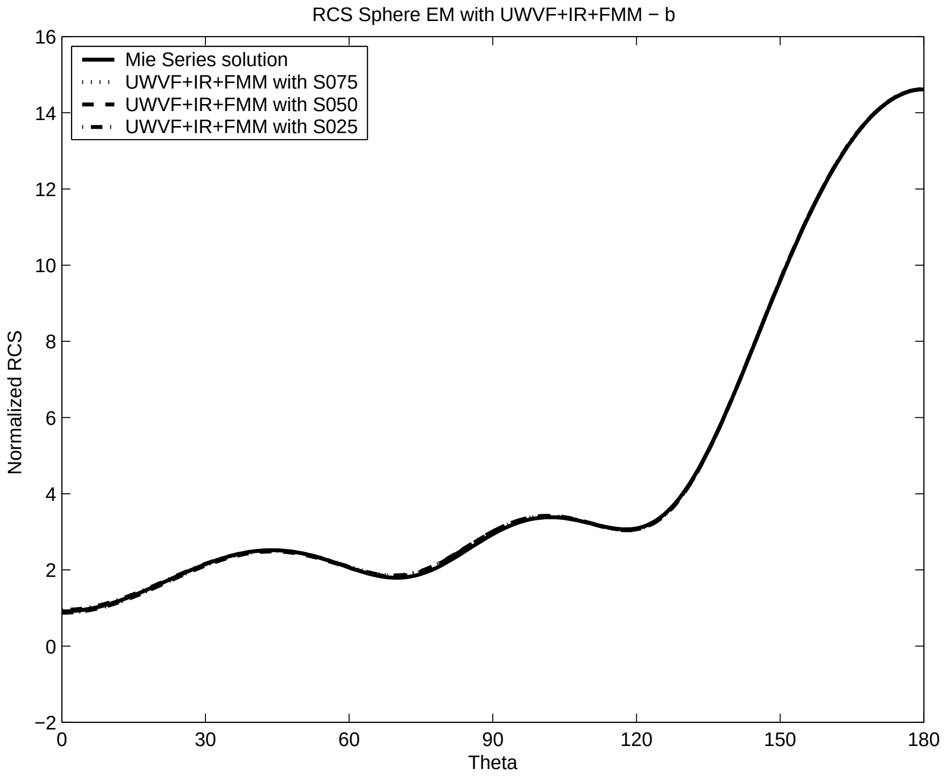
Classical UWVF with different meshes - EM polarisation



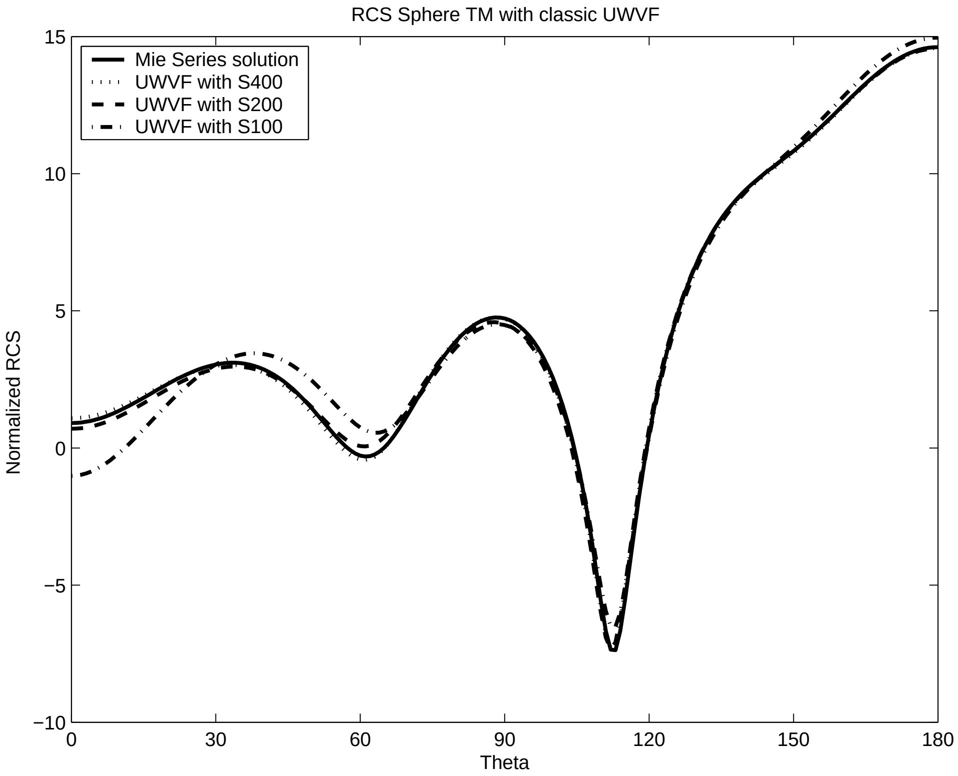


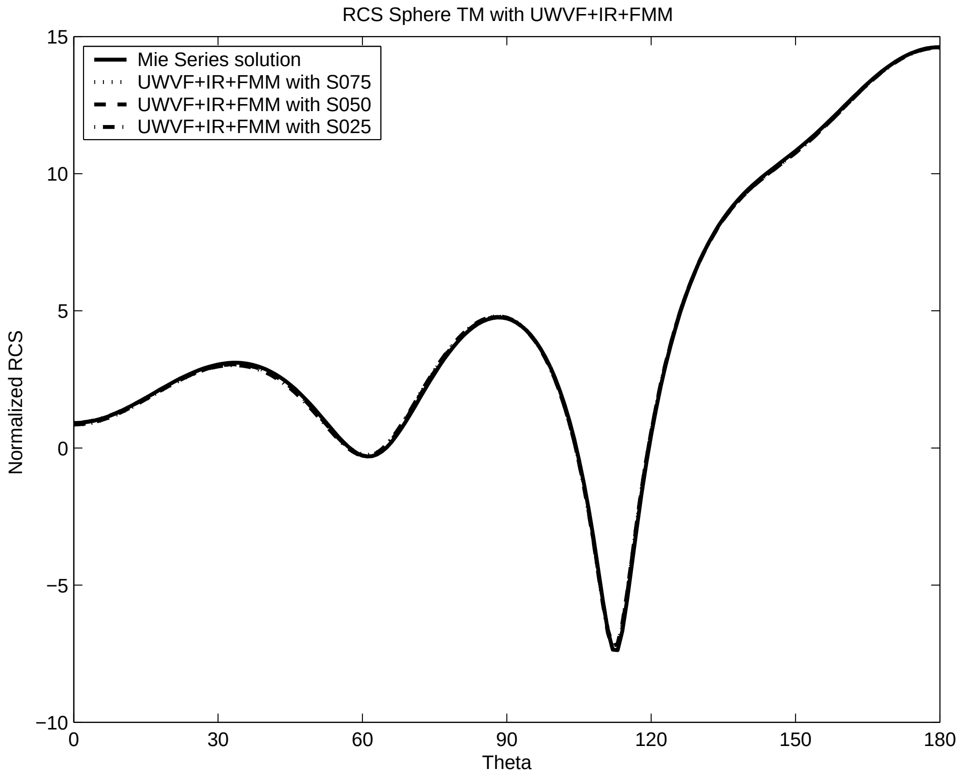
UWVF+IE+1LFMM with different meshes - EM polarisation





With different meshes - TM polarisation





Computational costs with UWVF (S400) and UWVF+IE+FMM (S025)

Case	T00	T0c	T0f	TD	TC	TCc	TCf	Tcg	Nit
S400	322	–	–	1.54	4.22	–	–	971	162
S025	15.5	394.5	13	0.13	0.4	13.6	10.5	1745	126

Case	Ttot	mem
S400	1293	4.8
S025	2168	2.1

Particularities of UWVF+IE+FMM:

- Integral Representation with no singularity
- Possibility of a FMM with no close interaction
  $\Longrightarrow$ No increase of the cost when local refinements

To be studied:

- Use of a MLFMM
- A double mesh on $\Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$