

**Ultra-Weak Variational Formulation
and Integral Representation
using a Fast Multipole Method
for the Equations of Electromagnetism**

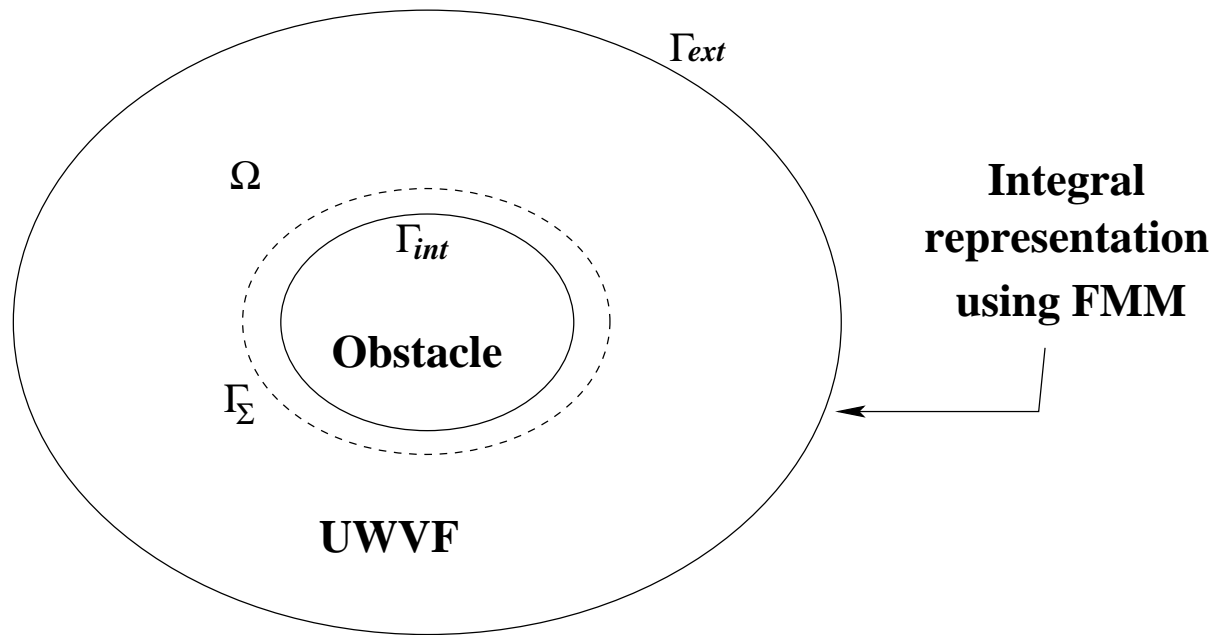
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Ω : “exterior” domain

Γ_{Σ} or Γ_{int} : Boundary of an obstacle

Γ_{ext} : Artificial boundary

Time-Harmonic Maxwell's Equations

For given m, j such that $\nabla \cdot m = 0 = \nabla \cdot j$, find (E, H) such that, in Ω

$$\left\{ \begin{array}{l} \nabla \wedge E - i\omega\mu H = -m, \\ \nabla \wedge H + i\omega\varepsilon E = j, \\ \nabla \cdot (\varepsilon E) = 0, \\ \nabla \cdot (\mu H) = 0, \end{array} \right.$$

with the Boundary Condition on $\Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$

$$- |\sqrt{\varepsilon}| E \wedge \nu + (|\sqrt{\mu}| H \wedge \nu) \wedge \nu = Q(|\sqrt{\varepsilon}| E \wedge \nu + (|\sqrt{\mu}| H \wedge \nu) \wedge \nu) + g ,$$

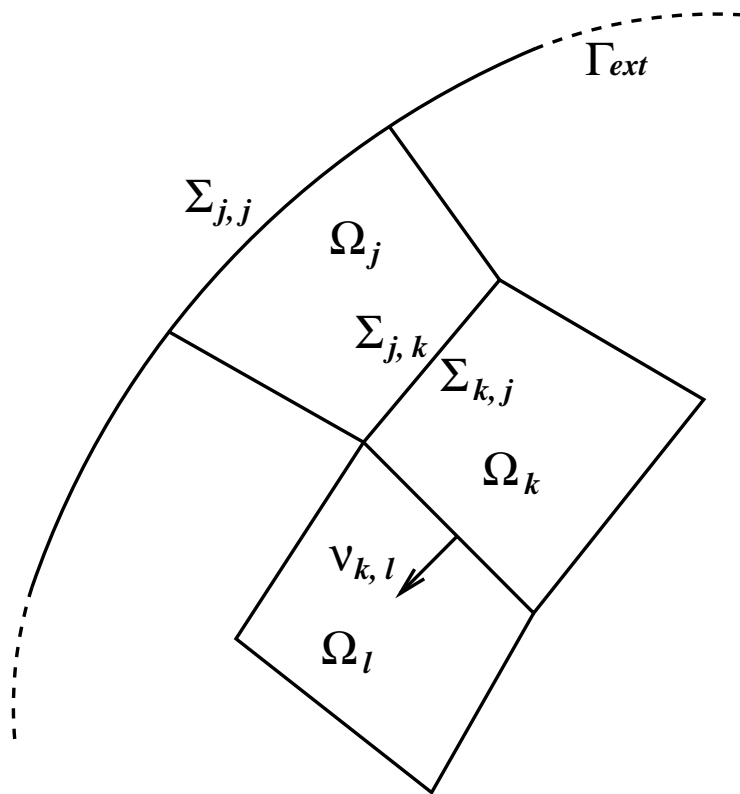
where

$Q = 0$ on Γ_{ext} (ABC) and g involving the incident wave.

Ultra-Weak Variational Formulation (UWVF)

O. CESSENAT - B. DESPRÉS

Decomposition of the domain: K sub-domains $\{\Omega_k\}_{k=1,\dots,K}$.



$$\partial\Omega_k = \bigcup_j \Sigma_{k,j}$$

$$\nu_k = \text{exterior normal to } \Omega_k$$

$$\nu_{kj} = \text{normal to } \Sigma_{k,j} \text{ from } \Omega_k \text{ to } \Omega_j$$

$$\varepsilon_k = \varepsilon_{/\Omega_k}$$

$$\varepsilon_{kj} = \left| \sqrt{\varepsilon_k \varepsilon_j} \right|$$

$$\varepsilon_{kk} = \left| \varepsilon_k \right|$$

Variational Space: $V = \prod_{k=1}^K L_t^2(\partial\Omega_k).$

Usual Scalar Product: $(\mathcal{X}, \mathcal{Y})_V = \sum_k \int_{\partial\Omega_k} \mathcal{X}_{/\partial\Omega_k} \overline{\mathcal{Y}_{/\partial\Omega_k}}.$

Assumption: ε and μ constant on each Ω_k , (E, H) found such that:

$$\begin{aligned} E_k, H_k &\in \tilde{\mathcal{H}}(\Omega_k, \partial\Omega_k) \\ &= \{F \in (H(\text{rot}, \Omega_k) \cap H(\text{div}_0, \Omega_k)); F_{/\partial\Omega_k} \wedge \nu_k \in L_t^2(\partial\Omega_k)\} \end{aligned}$$

The unknown of the UWVF: $\mathcal{X} \in V$ defined by $\mathcal{X}_{/\partial\Omega_k} = \mathcal{X}_k \in L_t^2(\partial\Omega_k)$
and

$$(\mathcal{X}_k)_{/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}(E_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H_k \wedge \nu_k) \wedge \nu_k)$$

Variational Formulation: Find $\mathcal{X} \in V$ such that

$$(\mathcal{X}, \mathcal{Y})_V - (\Pi\mathcal{X}, F\mathcal{Y})_V = (b, \mathcal{Y})_V$$

for all **test function** $\mathcal{Y} \in V$ defined by

$$(\mathcal{Y}_k)_{/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k)$$

such that $(E'_k, H'_k) \in (\tilde{\mathcal{H}}(\Omega_k, \partial\Omega_k))^2$ and

$$\begin{cases} \nabla \wedge E'_k - i\omega\bar{\mu}_k H'_k = 0, & \nabla \wedge H'_k + i\omega\bar{\varepsilon}_k E'_k = 0, \\ \sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k) \in L^2_t(\partial\Omega_k), \end{cases}$$

and with the **operators** $F : V \rightarrow V$ and $\Pi : V \rightarrow V$ defined by

$$(F\mathcal{Y})_{/\Sigma_{k,j}} = -\sqrt{\varepsilon_{kj}}(E'_k \wedge \nu_k) + \sqrt{\mu_{kj}}((H'_k \wedge \nu_k) \wedge \nu_k)$$

$$(\Pi\mathcal{Y})_{/\Sigma_{k,j}} = \mathcal{Y}_{/\Sigma_{j,k}} \quad , \quad (\Pi\mathcal{Y})_{/\Sigma_{k,k}} = Q\mathcal{Y}_{/\Sigma_{k,k}}$$

Galerkin Discretisation

Find $\mathcal{X}_h \in V_h$ such that $\forall \mathcal{Y}_h \in V_h$

$$(\mathcal{X}_h, \mathcal{Y}_h)_{V_h} - (\Pi \mathcal{X}_h, F \mathcal{Y}_h)_{V_h} = (b, \mathcal{Y}_h)_{V_h}$$

Basis Functions $Z_i, i \in J \rightarrow \mathcal{X}_h = \sum_{i \in J} X_i Z_i.$

Find $(X_i)_{i \in J} \in \mathbb{C}^{\#J}$ such that

$$(D - C)X = b$$

with

$$D_{kj} = \delta_{kj} (Z_j, Z_k)_{V_h} \quad \text{and} \quad C_{kj} = (\Pi Z_j, F Z_k)_{V_h}$$

Boundary condition included in C ($k = j$).

Choice of Z_i : On each element Ω_k

$$\text{for } l = 1, \dots, p, \quad Z_{kl/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}} E_{kl}^F \wedge \nu_k + \sqrt{\mu_{kj}} (H_{kl}^F \wedge \nu_k) \wedge \nu_k, \quad ,$$

$$\text{for } l = p + 1, \dots, 2p, \quad Z_{kl/\Sigma_{k,j}} = \sqrt{\varepsilon_{kj}} E_{kl}^G \wedge \nu_k + \sqrt{\mu_{kj}} (H_{kl}^G \wedge \nu_k) \wedge \nu_k, \quad ,$$

with

$$\begin{aligned} E_{kl}^F(X) &= \sqrt{\mu_{kj}} F_{kl} e^{i\omega\sqrt{\varepsilon_k\mu_k}(V_{kl}\cdot X)}, & H_{kl}^F(X) &= i\sqrt{\varepsilon_{kj}} F_{kl} e^{i\omega\sqrt{\varepsilon_k\mu_k}(V_{kl}\cdot X)}, \\ E_{kl}^G(X) &= \sqrt{\mu_{kj}} G_{kl} e^{i\omega\sqrt{\varepsilon_k\mu_k}(V_{kl}\cdot X)}, & H_{kl}^G(X) &= -i\sqrt{\varepsilon_{kj}} G_{kl} e^{i\omega\sqrt{\varepsilon_k\mu_k}(V_{kl}\cdot X)}. \end{aligned}$$

and

$$F_{kl} = E_{kl}^0 + i E_{kl}^0 \wedge V_{kl}, \quad G_{kl} = E_{kl}^0 - i E_{kl}^0 \wedge V_{kl}.$$

where $E_{kl}^0 \perp V_{kl}$

Integral Representation

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Previous ABC on the artificial boundary

$$-E \wedge \nu + (H \wedge \nu) \wedge \nu = -E_0 \wedge \nu + (H_0 \wedge \nu) \wedge \nu$$

Ideal Boundary Condition on Γ_{ext}

$$-E \wedge \nu + (H \wedge \nu) \wedge \nu = -E^s \wedge \nu + (H^s \wedge \nu) \wedge \nu - E_0 \wedge \nu + (H_0 \wedge \nu) \wedge \nu$$

On Γ_{ext} , (E^s, H^s) given by **the Stratton-Chu formulae**

$$E^s(x) = \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x, y) \nu_{\text{int}}(y) \wedge E(y) d\gamma(y) \\ - \frac{1}{i\omega} \nabla_x \wedge \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x, y) \nu_{\text{int}}(y) \wedge H(y) d\gamma(y) ,$$

$$\begin{aligned}
H^s(x) = & \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x, y) \nu_{\text{int}}(y) \wedge H(y) d\gamma(y) \\
& + \frac{1}{i\omega} \nabla_x \wedge \nabla_x \wedge \int_{\Gamma_{\text{int}}} G(x, y) \nu_{\text{int}}(y) \wedge E(y) d\gamma(y) \quad ,
\end{aligned}$$

On Γ_{int}

$$\nu \wedge E = \frac{Q-1}{2\sqrt{\epsilon_{kk}}} \mathcal{X} \quad \text{and} \quad \nu \wedge H = \frac{Q+1}{2\sqrt{\mu_{kk}}} \mathcal{X} \wedge \nu$$

That is

$$\nu_{\text{int}} \wedge E = -\frac{Q-1}{2\sqrt{\epsilon_{kk}}} \mathcal{X} \quad \text{and} \quad \nu_{\text{int}} \wedge H = -\frac{Q+1}{2\sqrt{\mu_{kk}}} \mathcal{X} \wedge \nu$$

The new discret system

$$(D - C - \tilde{C})X = b \quad , \quad \tilde{C} = \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 + \tilde{C}_4 \quad ,$$

with, for $i = 1, \dots, 4$, $\forall k, l$

$$(\tilde{C}_i \mathcal{X}_h)_{kl} = \int_{\Sigma_{kk}^{\text{ext}}} \frac{1}{\sqrt{\varepsilon_{kk} \mu_{kk}}} S_i(\mathcal{X}_h) \cdot \frac{1}{(-\sqrt{\varepsilon_{kk}} E'_{kl} \wedge \nu_k + \sqrt{\mu_{kk}} (H'_{kl} \wedge \nu_k) \wedge \nu_k)} d\gamma_{\text{ext}} \quad .$$

where

$$(S_2(\mathcal{X}))(x) = \frac{-1}{i\omega} \left(\omega^2 \int_{\Gamma_{\text{int}}} \frac{Q(y) + 1}{2 \sqrt{\mu(y)}} G(x, y) (\mathcal{X}(y) \wedge \nu(y)) d\gamma(y) - \int_{\Gamma_{\text{int}}} \frac{Q(y) + 1}{2 \sqrt{\mu(y)}} \nabla_x \nabla_y^t G(x, y) (\mathcal{X}(y) \wedge \nu(y)) d\gamma(y) \right) \wedge \nu(x) \quad ,$$

$$(S_1(\mathcal{X}))(x) = \left(- \int_{\Gamma_{\text{int}}} \frac{Q(y) - 1}{2 \sqrt{\varepsilon(y)}} \nabla_y G(x, y) \wedge \mathcal{X}(y) d\gamma(y) \right) \wedge \nu(x) ,$$

For instance

$$(\tilde{C}_1)_{kj}^{lm} = \int_{\Sigma_{kk}^{\text{ext}}} \frac{1}{\sqrt{\varepsilon_{kk} \mu_{kk}}} \left(- \int_{\Sigma_{jj}^{\text{int}}} \frac{Q_j - 1}{2 \sqrt{\varepsilon_{jj}}} \nabla_y G(x, y) \wedge Z_{jm}(y) d\gamma(y) \right) \wedge \nu(x) \cdot Z_{kl}(x) d\gamma(x) .$$

with

$$Z_{jm}(y) = \sqrt{\varepsilon_{jj}} E_{jm}^{F/G} \wedge \nu_j + \sqrt{\mu_{jj}} (H_{jm}^{F/G} \wedge \nu_j) \wedge \nu_j$$

$$Z_{kl}(x) = \overline{\left(-\sqrt{\varepsilon_{kk}} E_{kl}^{F/G} \wedge \nu_k + \sqrt{\mu_{kk}} (H_{kl}^{F/G} \wedge \nu_k) \wedge \nu_k \right)}$$

Numerical solution of $(D - C - \tilde{C})X = b$

Two ways:

- In the same way as previously with $C \leftarrow C + \tilde{C}$
- Another way: Iterative relaxed Jacobi method

$$X^{(n)} \leftarrow \alpha \tilde{X}^{(n)} + (1 - \alpha) X^{(n-1)}$$

with $\tilde{X}^{(n)}$ given by

$$(D - C)\tilde{X}^{(n)} = b + \tilde{C}X^{(n-1)}$$

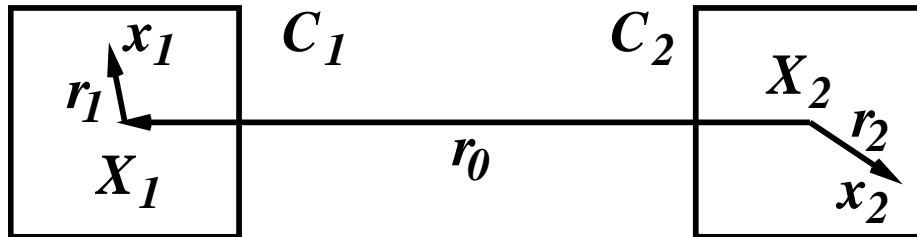
Algorithmic Complexity (1/2)

$p = \#$ of directions for the basis functions and $\kappa =$ wavenumber

Method	Number of elements	Cost of the solution
UWVF	κ^3	$\kappa^3 p^2$
UWVF+IE	κ^2	$\kappa^2 p^2 + \kappa^4 p^2$

Use of a FMM to reduce the cost related to the integral representation

FMM expansion



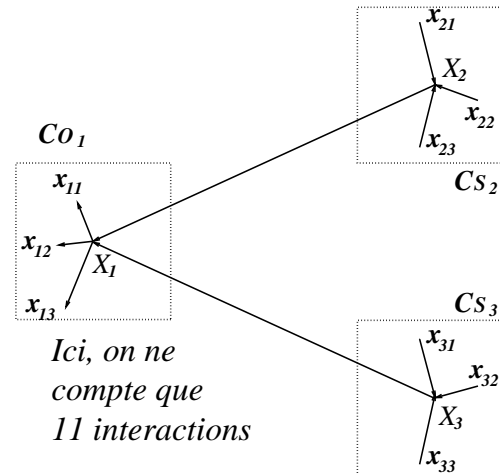
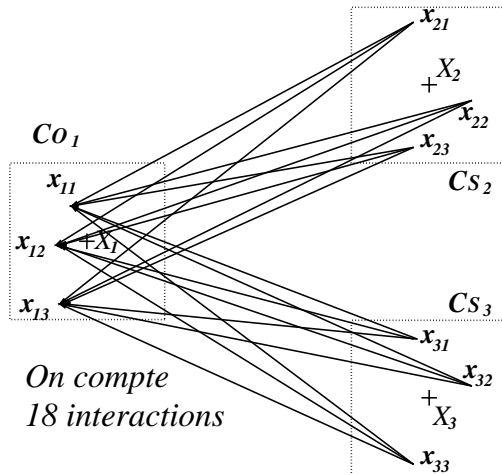
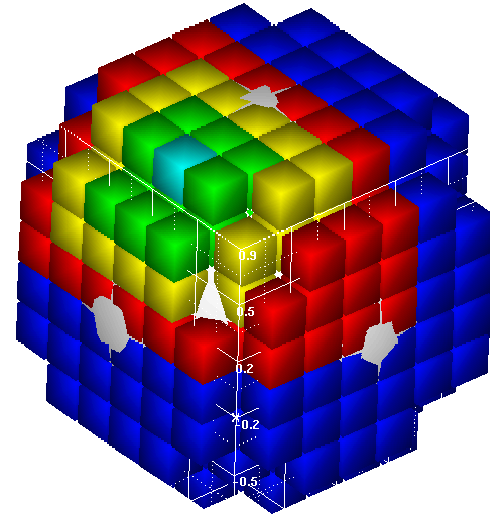
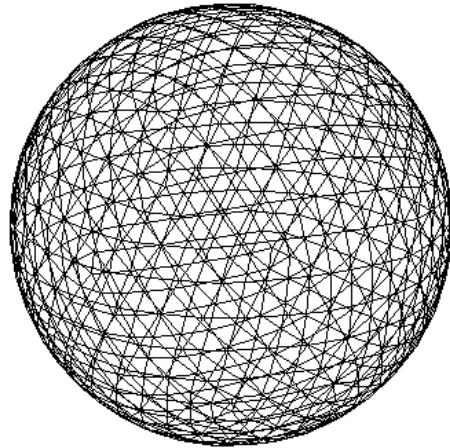
$$r_0 = X_1 - X_2 \quad r_i = x_i - X_i$$

$$r = r_1 - r_2 \quad |r| < |r_0|$$

$$G(x_1, x_2) = \frac{e^{i\kappa|x_1-x_2|}}{4\pi|x_1-x_2|} \approx \frac{1}{4\pi} \sum_{p=1}^S \omega_p e^{i\kappa s_p \cdot r_1} \mathcal{T}_{L,r_0}(s_p) e^{-i\kappa s_p \cdot r_2}$$

$$\mathcal{T}_{L,r_0}(s) = i\kappa \sum_{l=0}^L \frac{(2l+1)i^l}{4\pi} h_l^{(1)}(\kappa|r_0|) P_l(\cos(s, r_0))$$

- d = diameter of the boxes
- $L \sim \kappa d$ (Truncation parameter of the [Gegenbauer Series](#))
- $S \sim L^2$ (Discretization of the unit sphere in the [Funk-Hecke Formula](#))



Algorithmic Complexity (2/2)

$p = \#$ of directions for the basis functions and $\kappa =$ wavenumber

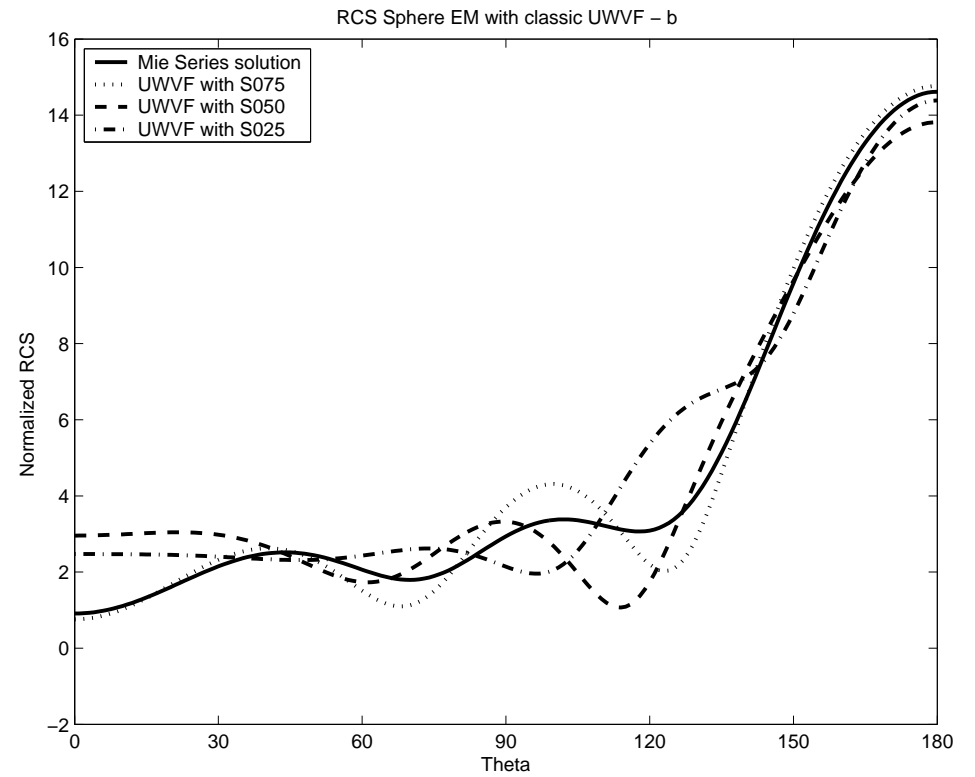
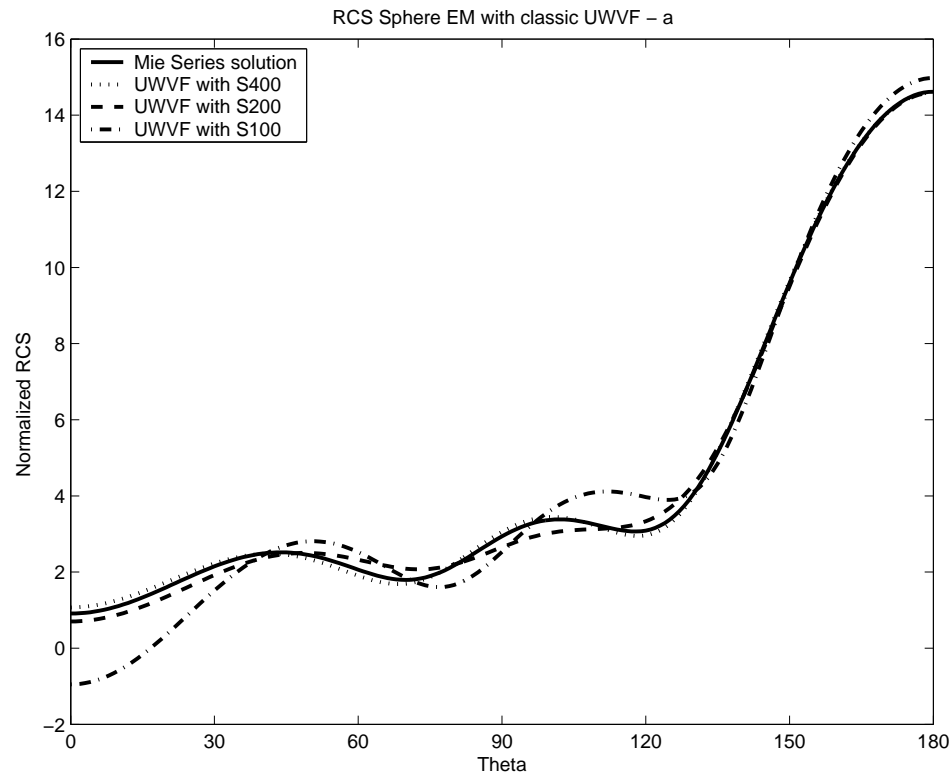
Method	Number of elements	Cost of the solution
UWVF	κ^3	$\kappa^3 p^2$
UWVF+IE	κ^2	$\kappa^2 p^2 + \kappa^4 p^2$
UWVF+IE+1LFMM	κ^2	$\kappa^2 p^2 + \kappa^3 p$
UWVF+IE+MLFMM	κ^2	$\kappa^2 p^2 + \kappa^2 \ln(\kappa) p$

Case of the unit sphere ; $\kappa = 4$; wavelength = $\lambda = \frac{\pi}{2}$

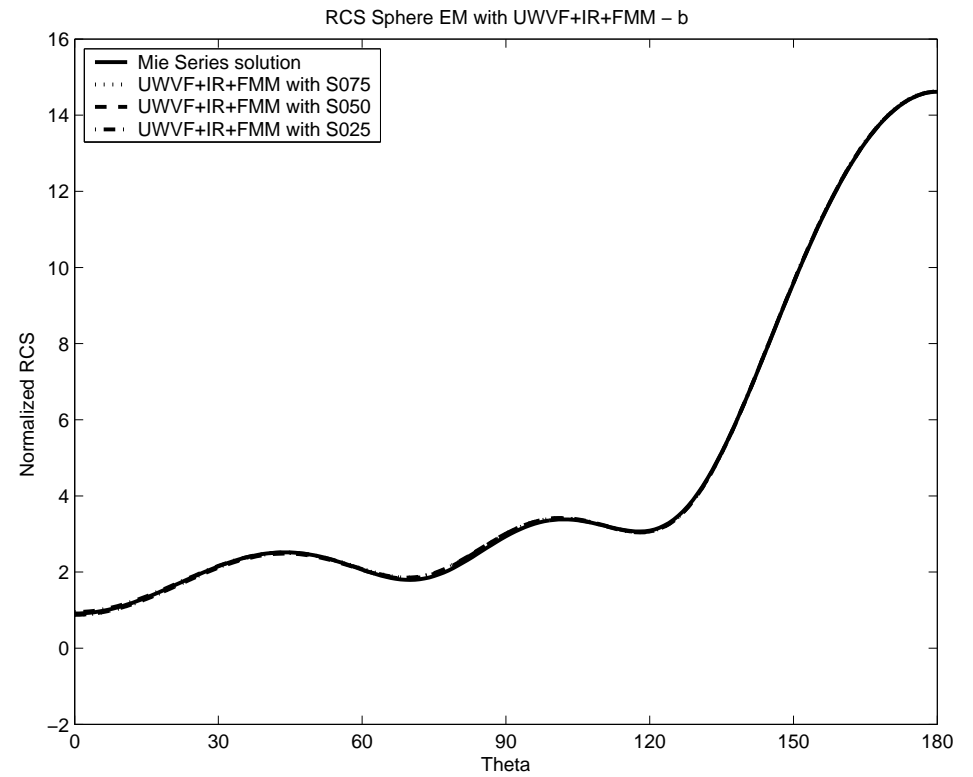
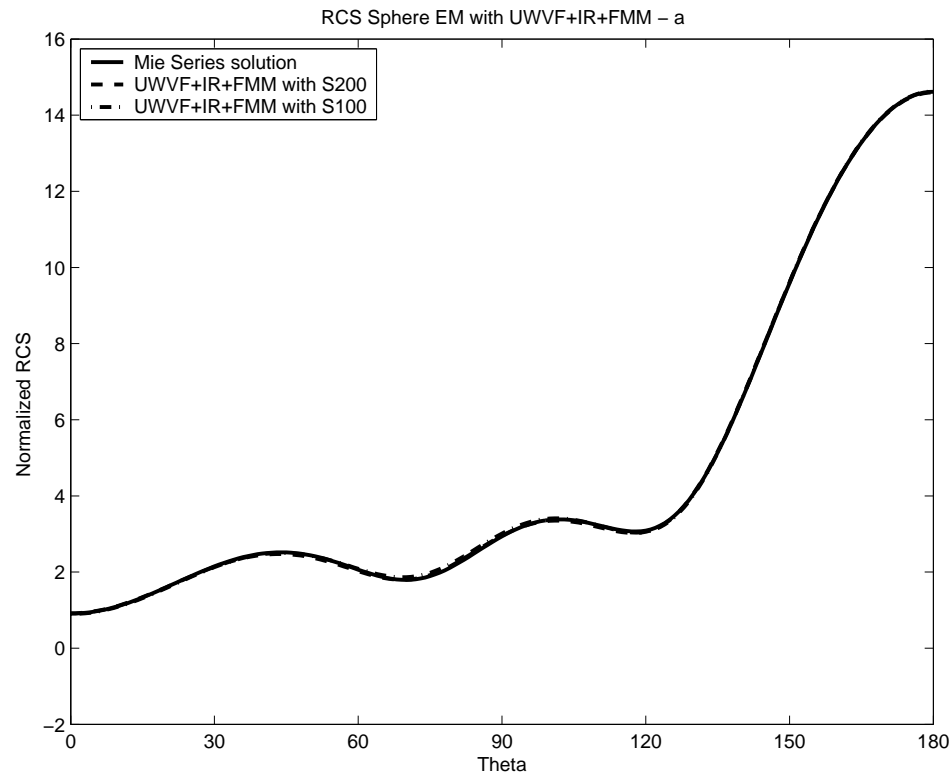
Artificial boundary at different distances (4 ; 2 ; 1 ; 0.75 ; 0.5 ; 0.25 m).

Name	S400	S200	S100	S075	S050	S025
Radius in m	5	3	2	1.75	1.5	1.25
Dist. between Γ_{int} and Γ_{ext}	$\approx 2.6\lambda$	$\approx 1.3\lambda$	$\approx 2\lambda/3$	$\approx \lambda/2$	$\approx \lambda/3$	$\approx \lambda/6$
Num. of tetrahedra	16179	14526	40609	30133	21083	11008
Num. of basis fcts per tetrahedron	8 to 128	8 to 72	10 to 32	10 to 30	8 to 28	10 to 24
Num. of DoF	880200	508450	753616	536874	356666	178146

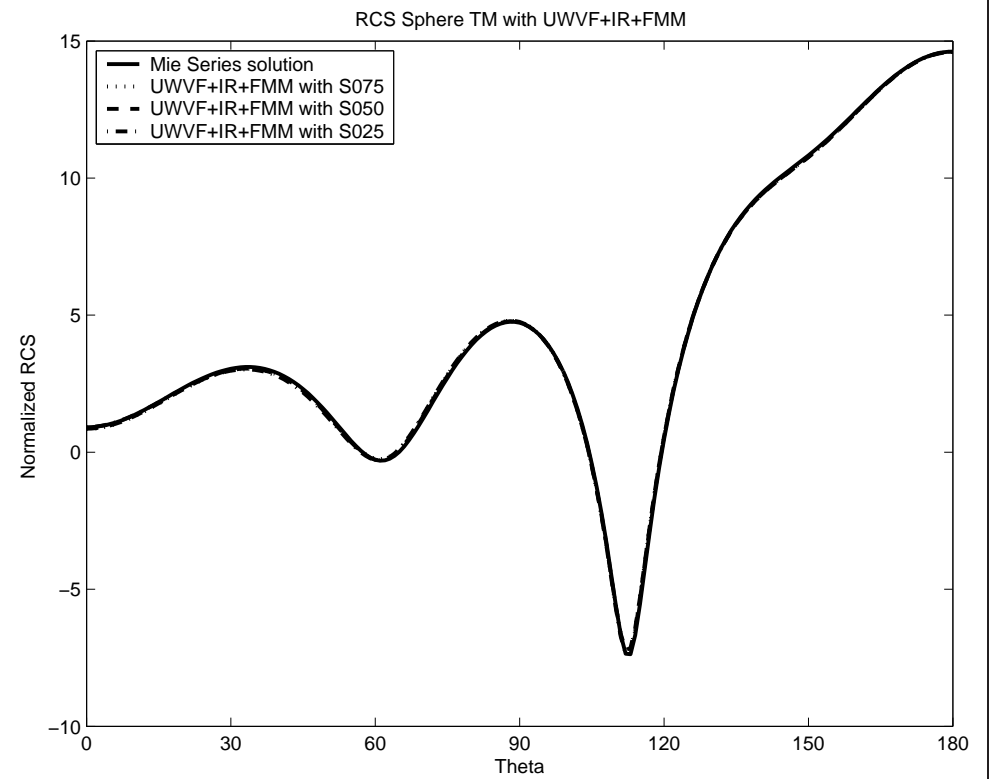
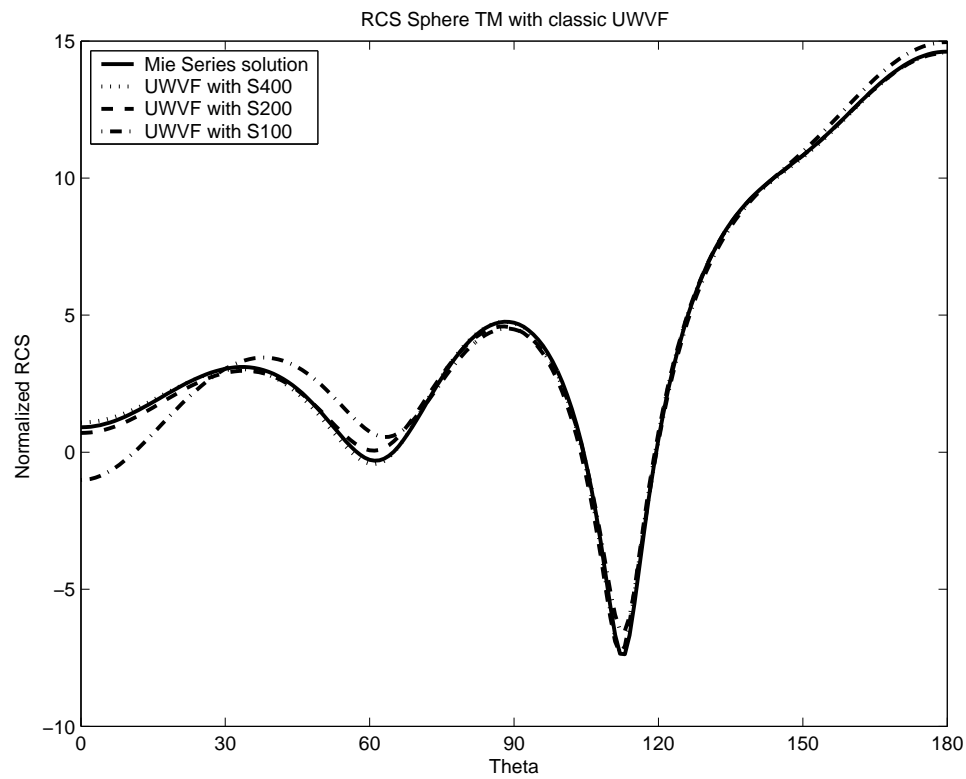
Classical UWVF with different meshes - EM polarisation



UWVF+IE+1LFMM with different meshes - EM polarisation



With different meshes - TM polarisation



Computational costs with UWVF (S400) and UWVF+IE+FMM (S025)

Case	T00	T0c	T0f	TD	TC	TCc	TCf	Tcg	Nit
S400	322	–	–	1.54	4.22	–	–	971	162
S025	15.5	394.5	13	0.13	0.4	13.6	10.5	1745	126

Case	Ttot	mem
S400	1293	4.8
S025	2168	2.1

Particularities of UWVF+IE+FMM:

- Integral Representation with no singularity
- Possibility of a FMM with no close interaction
⇒ No increase of the cost when local refinements

To be studied:

- Use of a MLFMM
- A double mesh on $\Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$