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Discontinuous Galerkin Methods as Weighted Residuals

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From joint works with
Arnold, Cockburn, Hughes, Marini, Masud, Süli,....

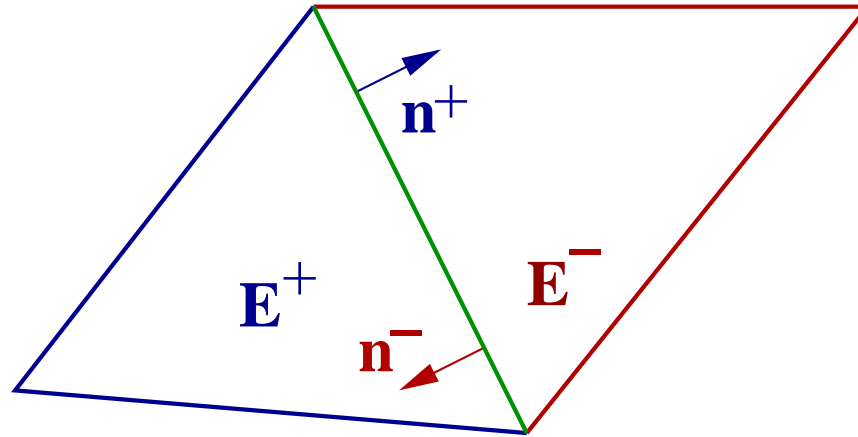
For Michel Crouzeix — Guidel, 2-3 Juin, 2006

PLAN:

- Original Derivation of DG Methods
- The Weighted Residuals Formulation
- Problems in mixed form
- Some numerical Results

AVERAGES AND JUMPS

\mathcal{T}_h : decomposition of Ω in elements K ; \mathcal{E}_h =edges of \mathcal{T}_h .



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \quad [v] = v^+ \mathbf{n}^+ + v^- \mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \quad [[\boldsymbol{\tau}]] = \boldsymbol{\tau}^+ \mathbf{n}^+ + \boldsymbol{\tau}^- \mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

On the boundary edges: $[v] = v\mathbf{n}$; $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

A MAGIC FORMULA

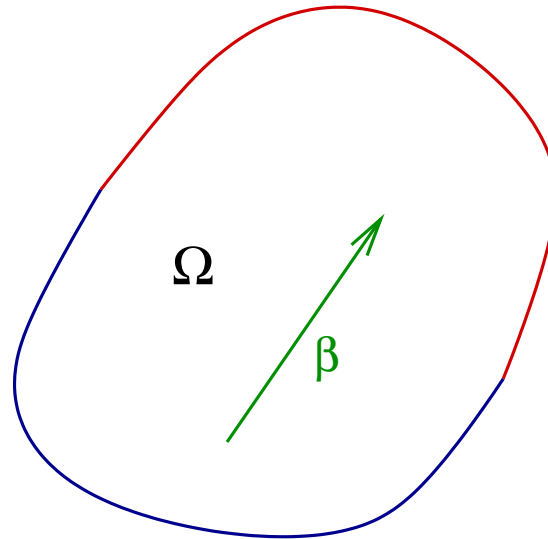
Assume that q is an edge-wise smooth scalar, and $\boldsymbol{\tau}$ an edge-wise smooth vector. We obviously accept that they have different values on the two sides of the same edge. Then, using the above definitions of jump $[\![\cdot]\!]$ and average $\{\cdot\}$, we have

$$\sum_K \int_{\partial K} q \boldsymbol{\tau} \cdot \mathbf{n} \, ds = \sum_e \int_e [\![q]\!] \cdot \{\boldsymbol{\tau}\} \, ds + \sum_{e'} \int_{e'} \{q\} [\![\boldsymbol{\tau}]\!] \, ds$$

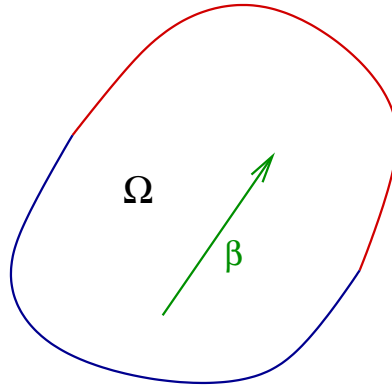
where e ranges over all edges and e' ranges over internal edges.

Clearly, there is **nothing** magic or deep: just *reordering terms in the sum*. But it is surely **nice**.

A HYPERBOLIC MODEL PROBLEM



Let Ω be a bounded polygonal domain in \mathbf{R}^2 , and let the advective velocity field $\beta = (\beta_1, \beta_2)^T$ be a vector-valued function defined on $\bar{\Omega}$ with $\beta_i \in C^1(\bar{\Omega})$, $i = 1, 2$.

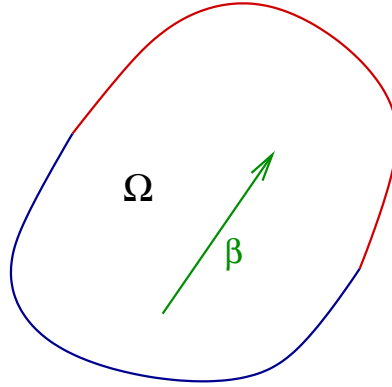


We define the *inflow* and *outflow* parts of $\Gamma = \partial\Omega$ in the usual fashion:

$$\Gamma_- = \{x \in \Gamma : \beta(x) \cdot \mathbf{n}(x) < 0\} = \text{inflow},$$

$$\Gamma_+ = \{x \in \Gamma : \beta(x) \cdot \mathbf{n}(x) > 0\} = \text{outflow},$$

where $\mathbf{n}(x)$ denotes the unit **outward** normal vector to Γ at $x \in \Gamma$.



Let moreover $\gamma \in C(\bar{\Omega})$ be the reactive term, $f \in L^2(\Omega)$ be the external source, and $g \in L^2(\Gamma_-)$ be the Dirichlet datum.

As a model problem we will consider the hyperbolic boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv \operatorname{div}(\beta u) + \gamma u &= f && \text{in } \Omega, \\ u &= g && \text{on } \Gamma_- \end{aligned} \tag{1}$$

Let us see the DG formulation (Lesaint-Raviart, Reed-Hill).

ORIGINAL DERIVATION OF THE DG METHOD

Multiply **the equation** by a test function v_h , and integrate over Ω :

$$\int_{\Omega} (\operatorname{div}(\boldsymbol{\beta}u) + \gamma u)v_h dx = \int_{\Omega} f v_h dx.$$

Then integrate by parts the first term:

$$\sum_{K \in \mathcal{T}_h} \int_K (-u (\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h) dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\boldsymbol{\beta} \cdot \mathbf{n}) u v_h ds = \int_{\Omega} f v_h dx.$$

Using the **magic formula**, neglecting $[[\boldsymbol{\beta}u]]$, and using $u = g$ on Γ_- :

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K (-u (\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h) dx \\ & + \sum_{e \notin \Gamma_-} \int_e \{\boldsymbol{\beta}u\} \cdot [v_h] ds + \sum_{e \subset \Gamma_-} \int_e \boldsymbol{\beta} \cdot \mathbf{n} g v_h ds = \int_{\Omega} f v_h dx. \end{aligned}$$

Then you write the equation putting u_h instead of u

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K (-u_h (\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u_h v_h) dx \\ + \sum_{e \notin \Gamma_-} \int_e \{\boldsymbol{\beta} u_h\} \cdot [v_h] ds + \sum_{e \subset \Gamma_-} \int_e \boldsymbol{\beta} \cdot \mathbf{n} g v_h ds = \int_{\Omega} f v_h dx. \end{aligned}$$

Finally you substitute the *average* $\{\boldsymbol{\beta} u_h\}$ by the *upwind average* $\{\boldsymbol{\beta} u\}_{up}$, defined, on every edge, as the value of u_h **coming from the upwind triangle**:

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K (-u_h (\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u_h v_h) dx \\ + \sum_{e \notin \Gamma_-} \int_e \{\boldsymbol{\beta} u_h\}_{up} \cdot [v_h] ds + \sum_{e \subset \Gamma_-} \int_e \boldsymbol{\beta} \cdot \mathbf{n} g v_h ds = \int_{\Omega} f v_h dx. \end{aligned}$$

AN ELLIPTIC MODEL PROBLEM

Let $\kappa \in \mathcal{L}^\infty(\Omega)$ be the diffusion coefficient, and consider the problem:

$$\begin{cases} Au \equiv -\operatorname{div}(\kappa \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

It is often convenient to introduce the flux $\boldsymbol{\sigma} = -\kappa \nabla u$ so that the problem splits in two equations

$$\begin{cases} \kappa^{-1} \boldsymbol{\sigma} + \nabla u = 0 & \text{in } \Omega \\ \operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Let us see its DG formulation (Arnold, Wheeler, Douglas-Dupont)

You multiply the equation by a test function $v \equiv v_h$ and integrate over Ω . Then integrate by parts.

$$\int_{\Omega} \kappa \nabla u \cdot \nabla_h v dx - \sum_K \int_{\partial K} \{\kappa \nabla u\} \cdot \mathbf{n} v_h ds = \int_{\Omega} f v dx.$$

Rearranging terms with the [magic formula](#) you have

$$\int_{\Omega} \kappa \nabla u \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla u\} ds - \sum_{e'} \int_{e'} [\kappa \nabla u] \cdot \{v\} ds = \int_{\Omega} f v dx.$$

Now you say: "Oh, but I know that u is smooth: hence $[\kappa \nabla u]$ is zero and I can forget about it!" You then write

$$\int_{\Omega} \kappa \nabla u \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla u\} ds = \int_{\Omega} f v dx.$$

You had

$$\int_{\Omega} \kappa \nabla u \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla u\} ds = \int_{\Omega} f v dx.$$

Then you say: "Since u is smooth, then also $[u] = 0$!" And you add a term *to restore symmetry*

$$\int_{\Omega} \kappa \nabla u \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla u\} ds - \sum_e \int_e [u] \cdot \{\kappa \nabla_h v\} ds = \int_{\Omega} f v dx.$$

Then you write u_h in place of u

$$\int_{\Omega} \kappa \nabla_h u_h \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla_h u_h\} ds - \sum_e \int_e [u_h] \cdot \{\kappa \nabla_h v\} ds = \int_{\Omega} f v dx.$$

Your discrete problem is now

$$\int_{\Omega} \kappa \nabla_h u_h \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\kappa \nabla_h u_h\} ds - \sum_e \int_e [u_h] \cdot \{\kappa \nabla_h v\} ds = \int_{\Omega} f v dx.$$

Then you say: "Gosh! My method is unstable! However, since $[u] = 0$, I can add a stabilizing term"

$$\begin{aligned} \int_{\Omega} \nabla_h u_h \cdot \nabla_h v dx - \sum_e \int_e [v] \cdot \{\nabla_h u_h\} ds - \sum_e \int_e [u_h] \cdot \{\nabla_h v\} ds \\ + \sum_e \frac{\gamma}{|e|} \int_e [u_h] \cdot [v] ds = \int_{\Omega} f v dx. \end{aligned}$$

And you are happy. This is "IP" (Interior Penalty).

DG MIXED FORMULATION

Let us see the DG mixed formulation (Bassi-Rebay, Cockburn-Shu).

You multiply the equations $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ and $\operatorname{div}\boldsymbol{\sigma} = f$ by the test functions $\boldsymbol{\tau}$ and v , respectively, and integrate by parts

$$\begin{aligned} \int_{\Omega} \kappa^{-1} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \sum_K \int_K u \operatorname{div} \boldsymbol{\tau} dx + \sum_K \int_K \hat{u} \boldsymbol{\tau} \cdot \mathbf{n} ds = 0 \\ - \sum_K \int_K \boldsymbol{\sigma} \cdot \nabla v dx + \sum_K \int_K v \hat{\boldsymbol{\sigma}} \cdot \mathbf{n} ds = \int_{\Omega} f v \end{aligned}$$

where \hat{u} and $\hat{\boldsymbol{\sigma}}$ are the **numerical fluxes** meant to approximate $u|_{\partial E}$ and $\boldsymbol{\sigma}|_{\partial E} \equiv -\kappa \nabla u|_{\partial E}$ (respectively), **to be modelled later on**.

Then you apply the **magic formula** assuming that \hat{u} and $\hat{\sigma}$ are single valued

$$\int_{\Omega} \kappa^{-1} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_{\Omega} u \operatorname{div}_h \boldsymbol{\tau} dx + \sum_{e'} \int_{e'} \hat{u} [\boldsymbol{\tau}] ds = 0$$

$$- \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla_h v dx + \sum_e \int_e \hat{\sigma} [v] ds = \int_{\Omega} f v$$

Then you put $\boldsymbol{\sigma}_h$ in place of $\boldsymbol{\sigma}$ and u_h in place of u , and you integrate by parts back (!) the first equation

$$\int_{\Omega} \kappa^{-1} \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_{\Omega} \nabla_h u_h \cdot \boldsymbol{\tau} dx + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\boldsymbol{\tau}] ds + \sum_e \int_e [u_h] \{\boldsymbol{\tau}\} ds = 0$$

$$- \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla_h v dx + \sum_e \int_e \hat{\sigma} [v] ds = \int_{\Omega} f v$$

If your choice of \hat{u} depends only on u_h , and if the gradients of the discretized scalars are contained in the space of discretized vectors, you can use the first equation

$$\int_{\Omega} \kappa^{-1} \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_{\Omega} \nabla u_h \cdot \boldsymbol{\tau} dx + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\boldsymbol{\tau}] ds - \sum_e \int_e [u_h] \{\boldsymbol{\tau}\} ds = 0$$

to express $\boldsymbol{\sigma}_h$ directly as an explicit function of u_h

$$\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h(u_h)$$

and substitute in the second

$$- \int_{\Omega} \boldsymbol{\sigma}(u_h) \cdot \nabla_h v dx + \sum_e \int_e \hat{\boldsymbol{\sigma}} [v] ds = \int_{\Omega} f v.$$

For various choices of \hat{u} and $\hat{\boldsymbol{\sigma}}$ you get a whole ZOO of methods (Arnold-B-Cockburn-Marini).

THE NEW APPROACH (B-Cockburn-Marini-Süli)

Let us see the basic ideas behind it. We consider a rather general system of PDE equations in a domain Ω and on its boundary $\partial\Omega$, that we write as

$$\mathcal{A}(\mathbf{u}) = 0.$$

Here the (nonlinear) operator \mathcal{A} includes the right-hand side as well as the boundary conditions, so that, say, the problem

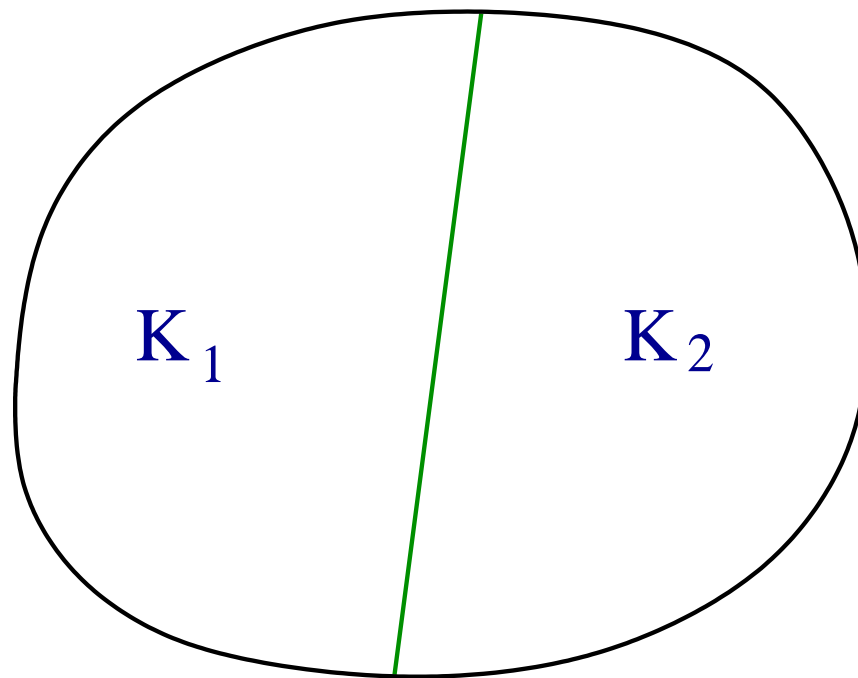
$$-\Delta u = f \text{ in } \Omega \quad \text{with} \quad u = g \text{ on } \partial\Omega,$$

would correspond to the operator

$$\mathcal{A}(u) := (\Delta u + f, \gamma_0 u - g)$$

in suitable functional spaces, where γ_0 is the *trace operator* on $\partial\Omega$.

Assume now that you look, *a priori* for a solution \mathbf{u} that is (reasonably) smooth separately in two (or more) subdomains K_1, \dots, K_k with $\Omega = \overline{\bigcup_j K_j}$.



If we want to write that such a \mathbf{u} is a solution of our original problem $\mathcal{A}(\mathbf{u}) = 0$, we have now to take into account:

- i) the **residual** of the partial differential equation(s) in each element,
- ii) the **residual** in the boundary conditions on $\partial\Omega$, *and*
- iii) the jumps of suitable trace operators applied to \mathbf{u} at interfaces (that we consider as **residuals** of suitable *continuity conditions*).

We indicate all these residuals by $R_1(\mathbf{u}), R_2(\mathbf{u}), \dots, R_n(\mathbf{u})$. The original problem $\mathcal{A}(\mathbf{u}) = 0$ is then changed into

$$R_i(\mathbf{u}) = 0 \quad (i = 1, \dots, n)$$

For instance, for Laplace operator in two subdomains we have

$$\begin{aligned} \Delta u + f &= 0 \text{ in } \Omega_i & \gamma_0 u - g &= 0 \text{ on } \partial\Omega \cap \partial\Omega_i & (i = 1, 2) \\ [u] &= 0 & [\nabla_h u] &= 0 & \text{on } \partial\Omega_1 \cap \partial\Omega_2 \end{aligned}$$

We consider now a decomposition of Ω into polygonal *elements* K_j . In order to proceed toward a discretized problem we would like to impose $R_i(\mathbf{u}) = 0$, $i = 1, 2, \dots, n$ in some weak (variational) sense. For this we consider a functional space \mathcal{V} with the following properties.

- The functions in \mathcal{V} can be discontinuous from one element K_j to another.
- The residual operators $R_1(\mathbf{u}), R_2(\mathbf{u}), \dots, R_n(\mathbf{u})$ map \mathcal{V} either into $L^2(K_j)$ (for some element K_j) or to $L^2(e)$ (for some edge e).
- The exact solution \mathbf{u} belongs to \mathcal{V} .

Then we define suitable **weight operators** $W_1(\mathbf{v}), W_2(\mathbf{v}), \dots, W_n(\mathbf{v})$ and we consider the variational problem :

Find $\mathbf{u} \in \mathcal{V}$ *such that*

$$\sum_{i=1}^n (R_i(\mathbf{u}), W_i(\mathbf{v}))_i = 0 \quad \forall \mathbf{v} \in \mathcal{V},$$

where the $(\cdot, \cdot)_i$'s are suitable (L^2 -type) inner products (either inside each element, or on the boundaries of the elements). A sufficient condition for having a unique solution is that the **W_i 's** map \mathcal{V} into a **dense subset** of the corresponding product of L^2 spaces (on elements, and on edges). We shall assume that this is verified.

In particular, you cannot allow any of the W_i to be zero.

We choose now a finite element space V_h of piecewise polynomial functions. If \mathcal{V} is not *totally crazy*, we will have $V_h \subset \mathcal{V}$.

The corresponding discrete problem will than be:

Find $\mathbf{u}_h \in V_h$ such that

$$\sum_{i=1}^n (R_i(\mathbf{u}_h), W_i(\mathbf{v}_h))_i = 0 \quad \forall \mathbf{v}_h \in V_h.$$

We point out explicitly that the existence and uniqueness that we have for the continuous problem will **not** be inherited by the discrete problem, unless the choices of the weights W_i and of the space V_h are *smart enough*. *No free lunch. Sorry!*

This approach is quite general and it applies to a number of different problems. It is clear, however, that the stability and the accuracy of the resulting scheme will depend heavily on the choice of the weights $W_1(\mathbf{v}_h), W_2(\mathbf{v}_h), \dots, W_n(\mathbf{v}_h)$. Suitable choices of the weights can also help in getting a *nicer* formulation (e.g. symmetric, when the problem is itself symmetric, easier to implement, and so on). However in order to reach a stable method it is, in general, necessary to add a *least-square part* to them. Roughly speaking, this means choosing *at least one* of the W_i 's of the form

$$W_i(\mathbf{v}_h) = N_i(\mathbf{v}_h) + \mu_i R_i(\mathbf{v}_h)$$

where $N_i(\mathbf{v}_h)$ (possibly equal to zero) is the part used to make the method *nicer*, while μ_i is a suitable (in general, mesh dependent) coefficient, and the term $\mu_i R_i(\mathbf{v}_h)$ is there to *stabilize* the method.

WEIGHTED RESIDUALS FORMULATION FOR DARCY

Coming back to the Darcy problem $-\operatorname{div} \kappa \nabla u = f$, with zero Dirichlet boundary conditions, our residual equations are now:

- $-\operatorname{div} \kappa \nabla u - f = 0$ in each element
- $[[u]] = 0$ on each edge
- $[[\boldsymbol{\sigma}]] = 0$ on each internal edge

(remember that $\boldsymbol{\sigma} = -\kappa \nabla u$).

A HOMEMADE ESTIMATE

It is not difficult to see that the residual equations above are *equivalent* to the original problem.

For instance if w is a (discontinuous) function, belonging to $H^2(K)$ for each K , then (setting $Aw := -\operatorname{div} \kappa \nabla w$)

$$\|w\|_{L^2(\Omega)}^2 \leq C \left(\sum_{K \in \mathcal{T}_h} \|Aw\|_{L^2(K)}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|[w]\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_h^\circ} \frac{1}{|e|} \|\llbracket \kappa \nabla w \rrbracket\|_{L^2(e)}^2 \right)$$

But, surely, the "a posteriori guys" know better ;-))

THE STARTING POINT

We choose $\mathcal{V} = H^2(\mathcal{T}_h)$ as the set of functions that are in H^2 of each element, separately. We group the W_i in three operators B_0 , B_1 , B_2 from $H^2(\mathcal{T}_h)$ to $L^2(\mathcal{T}_h)$, $L^2(\mathcal{E}_h)$ and $L^2(\mathcal{E}_h^\circ)$ respectively.

Then we consider the following *variational* formulation

Find $u \in H^2(\mathcal{T}_h)$ such that

$$(Au - f, B_0(v))_h + \langle [u], B_1(v) \rangle_h + \langle [[\sigma]], B_2(v) \rangle_h^0 = 0$$

for all $v \in H^2(\mathcal{T}_h)$, where

$$(u, v)_h = \sum_{K \in \mathcal{T}_h} \int_K u v \, dx \quad \langle u, v \rangle_h = \sum_{e \in \mathcal{E}_h} \int_e u v \, ds$$

and $\langle u, v \rangle_h^0$ runs only on internal edges

SUFFICIENT CONDITIONS ON THE OPERATORS B_j

$$(Au - f, B_0(v))_h + \langle [u], B_1(v) \rangle_h + \langle [\sigma], B_2(v) \rangle_h^0 = 0 \quad \forall v \in \mathcal{V}$$

The above equation gives back the original three equations on u (that is: $Au = f$, $[u] = 0$, and $[\sigma] = 0$) **if**

- $\forall K \in \mathcal{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathcal{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \psi \in C_0^\infty(e)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_1 v = \psi \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

CHOICE OF B_0

- Choosing $B_0 v \equiv v$ and using the magic formula, we can write:

$$\begin{aligned} (Au, B_0 v)_h &\equiv \sum_{K \in \mathcal{T}_h} \int_K -\operatorname{div}(\kappa \nabla u) v \, dx \\ &= \sum_{K \in \mathcal{T}_h} \left\{ \int_K \kappa \nabla u \cdot \nabla v \, dx + \int_{\partial K} \boldsymbol{\sigma} \cdot \mathbf{n}_K v \, ds \right\} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \kappa \nabla u \cdot \nabla v \, dx + \sum_e \int_e [v] \cdot \{\boldsymbol{\sigma}\} \, ds + \sum_{e'} \int_{e'} \{v\} [\boldsymbol{\sigma}] \, ds \\ &= (\kappa \nabla u, \nabla v)_h + \langle \{\boldsymbol{\sigma}\}, [v] \rangle_h + \langle [\boldsymbol{\sigma}], \{v\} \rangle_h^0. \end{aligned}$$

CHOICE OF B_2

$$(Au - f, B_0(v))_h + \langle [u], B_1(v) \rangle_h + \langle [\sigma], B_2(v) \rangle_h^0 = 0 \quad \forall v \in \mathcal{V}$$

With the previous choice of B_0 we had

$$(Au, B_0 v)_h = (\kappa \nabla u, \nabla v)_h + \langle \{\sigma\}, [v] \rangle_h + \langle [\sigma], \{v\} \rangle_h^0$$

- Choosing $B_2(v) \equiv -\{v\}$ gives

$$\begin{aligned} (Au - f, B_0(v))_h + \langle [u], B_1(v) \rangle_h + \langle [\sigma], B_2(v) \rangle_h^0 \\ \equiv (\kappa \nabla u, \nabla v)_h - \langle \{\kappa \nabla u\}, [v] \rangle_h - (f, v)_h + \langle [u], B_1(v) \rangle_h \end{aligned}$$

and we have just to choose B_1

CHOICES FOR B_1

Our equations are

$$(\kappa \nabla u, \nabla v)_h - \langle \{\kappa \nabla u\}, [v] \rangle_h + \langle [u], B_1(v) \rangle_h = (f, v)_h$$

You are not allowed to take $B_1 \equiv 0$. But you can take, for instance

- $B_1(v)|_e := -\{\kappa \nabla_h v\} + \left(\frac{c}{|e|} [v]\right)$ (stabilized) IP

OR

- $B_1(v)|_e := +\{\kappa \nabla_h v\} + \left(\frac{c}{|e|} [v]\right)$ (stabilized) Baumann-Oden

OR

- $B_1(v)|_e := \frac{c}{|e|} [v]$ Wheeler, Sun

OTHER POSSIBILITIES

On the other hand, the choice of B_2 can also be revisited. For instance taking

$$B_2 v|_e = -\{v\} + \gamma|e|[\nabla_h v] \quad \text{and} \quad B_1(v)|_e := -\{\kappa \nabla_h v\} + \frac{c(\kappa)}{|e|}[v]$$

(Douglas-Dupont, Hansbo et als, etc.) your final bilinear form will be

$$\begin{aligned} & (\kappa \nabla u, \nabla v)_h - \langle \{\kappa \nabla u\}, [v] \rangle_h - \langle \{\kappa \nabla_h v\}, [u] \rangle_h \\ & + \sum_{e \in \mathcal{E}_h} \frac{c(\kappa)}{|e|} \int_e [u] \cdot [v] ds + \sum_{e \in \mathcal{E}_h^\circ} \gamma|e| \int_e [\kappa \nabla u] \cdot [\nabla_h v] ds \end{aligned}$$

ERROR ESTIMATES

$$V_h := \{v \in L^2(\Omega) : v|_K \in P_r(K) \forall K \in \mathcal{T}_h\} \subset H^2(\mathcal{T}_h)$$

The final bilinear form $a(u, v)$ should satisfy:

$$1) a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \quad 2) \alpha \|v_h\|^2 \leq a(v_h, v_h) \quad \forall v_h \in V_h$$

$$3) a(w, v) \leq C \|w\| \|v\| \quad \forall w, v \in H^2(\mathcal{T}_h) \quad 4) \|u - u_I\| \leq C h^r \|u\|_{r+1}$$

where

$$\|v\|^2 := \|\nabla_h v\|_0^2 + \sum_e |e|^{-1} \|[v]\|_{0,e}^2 + \sum_K h_K^2 |v|_{2,K}^2$$

and u_h (discrete solution), and u_I (interpolant of u) are in V_h . Then:

$$\begin{aligned} \alpha \|u_h - u_I\|^2 &\leq a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I) \\ &\leq C \|u - u_I\| \|u_h - u_I\| \leq C h^r \|u_h - u_I\|. \end{aligned}$$

MIXED DG FORMULATIONS FOR DARCY FLOWS

We consider again the mixed formulation for Darcy flow problem, this time with Neumann Boundary conditions

$$\begin{cases} \boldsymbol{\sigma} & = & -\kappa \nabla u & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} & = & f & \text{in } \Omega, \\ \boldsymbol{\sigma} \cdot \mathbf{n} & = & g & \text{on } \Gamma = \partial\Omega. \end{cases}$$

Note that now we have **Neumann boundary conditions** all over the boundary, instead of the previous Dirichlet ones. Hence we obviously have to require the **compatibility condition**:

$$\int_{\Omega} f dx = \int_{\Gamma} g ds$$

that will give us a unique solution $u \in H^1(\Omega)/\mathbb{R}$, (and a unique $\boldsymbol{\sigma} \in H(\operatorname{div}; \Omega)$).

THE DG RESIDUALS FOR MIXED FORMULATIONS

In view of using discontinuous elements, we have to enforce the following equations:

- $\kappa^{-1} \boldsymbol{\sigma} + \nabla u = 0$ in each element
- $\operatorname{div} \boldsymbol{\sigma} = f$ in each element
- $[[u]] = 0$ on each internal edge
- $[[\boldsymbol{\sigma}]] = 0$ on each internal edge
- $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on each boundary edge

COUPLING THE RESIDUALS IN TWO EQUATIONS

As we have **two unknowns**, it will be very convenient to group all the residuals in **two equations**. This can obviously be done **in a number of ways**.

For the sake of simplicity, we shall present just **one** way of grouping them, and to make thing further simple, we do it directly **at the discrete level**.

For this, for every decomposition \mathcal{T}_h of Ω into elements K , and for every couple of nonnegative integers ℓ and k , we introduce the finite element spaces

$$\Sigma_h^k = \{\boldsymbol{\tau} \in [L^2(\Omega)]^2 : \boldsymbol{\tau}|_K \in [P_k(K)]^2 \quad \forall K \in \mathcal{T}_h\}$$

$$V_h^\ell = \{v \in L^2(\Omega)/\mathbb{R} : v|_K \in P_\ell(K) \quad \forall K \in \mathcal{T}_h\}.$$

A REASONABLE COUPLING

We first consider the **first discretized equation**, where we group:
 $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ in K , $\forall K \in \mathcal{T}_h$ and $[[u]] = 0$ on e , $\forall e \in \mathcal{E}_h^\circ$ as follows:

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h))_h + \langle [[u_h]], B_1(\boldsymbol{\tau}_h) \rangle_h^0 = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k.$$

In the **second discretized equation** we take $\operatorname{div} \boldsymbol{\sigma} = f$ in $K \forall K \in \mathcal{T}_h$ together with $[[\boldsymbol{\sigma}]] = 0$ on e , $\forall e \in \mathcal{E}_h^\circ$, and $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on $\partial\Omega$. We then have:

$$\begin{aligned} & (\operatorname{div}_h \boldsymbol{\sigma}_h - f, B_2(v_h))_h + \langle [[\boldsymbol{\sigma}_h]], B_3(v_h) \rangle_h^0 \\ & \quad + \langle \boldsymbol{\sigma}_h \cdot \mathbf{n} - g, B_3(v_h) \rangle_h^\partial = 0 \quad \forall v_h \in V_h^\ell \end{aligned}$$

A ZOO

Changing the weights, we recover *an incredibly rich zoo*. The analysis, however, follows again *a common pattern*: after *eliminating* σ_h from the first equation one gets

$$B_h(u_h, v_h) = (f, v_h) - \langle g, v_h \rangle \quad \forall v_h \in V_h^\ell.$$

Setting $|||v|||^2 := |v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} ||[v]||_{0,e}^2$ we check the validity of the classical properties

- **Boundedness:** very easy in general

$$\begin{aligned} \exists C_b > 0 \text{ such that } B_h(u, v) &\leq C_b |||u||| |||v||| \quad \forall u, v \in V(h) \\ \text{where } V(h) &:= V_h^\ell + H^2(\Omega) \subset H^2(\mathcal{T}_h) \end{aligned}$$

- **Stability:** depending on the use of suitable *least square terms*

$$\exists C_s > 0 \text{ such that } B_h(v_h, v_h) \geq C_s |||v_h|||^2 \quad \forall v_h \in V_h^\ell$$

CONSISTENCY GIVES (MINOR) TROUBLES

Consistency always holds for the original mixed formulations. However, the elimination of $\boldsymbol{\sigma}_h$ from the [first equation](#)

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h))_h + \langle [u_h], B_1(\boldsymbol{\tau}_h) \rangle_h = 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k,$$

when you plug u in place of u_h , yields $\boldsymbol{\sigma}_h = -\kappa \Pi_{\boldsymbol{\Sigma}_h^k} \nabla u$. This, tested against $\nabla_h v_h$, [loses the \$\Pi_{\boldsymbol{\Sigma}_h^k}\$ only if \$\nabla_h V_h^\ell \subset \boldsymbol{\Sigma}_h^k\$](#) (that is, with our choices, [when \$k + 1 \geq \ell\$](#)).

As a consequence we usually have

$$B_h(u, v_h) = (f, v_h) - \langle g, v_h \rangle$$

[only if \$\nabla_h V_h^\ell \subset \boldsymbol{\Sigma}_h^k\$](#) . Otherwise we have [a consistency error](#) which depends on mismatch between the spaces V_h^ℓ and $\boldsymbol{\Sigma}_h^k$. This however can be dealt with rather easily.

BACK TO STABILITY

Stability is not difficult to enforce *acting on the jump terms* (jump stabilization) *or on the weights inside the elements*, as for instance using a so-called Hughes-Franca stabilization of the type

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h, v_h))_h := (\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \boldsymbol{\tau}_h + \theta(\boldsymbol{\tau}_h + \kappa \nabla v_h))_h.$$

It is remarkable that the effects of a Hughes-Franca stabilization are nearly identical to the effects of the jump stabilization. On the other hand, the jump stabilization, in this context, is itself a Hughes - Franca stabilization.

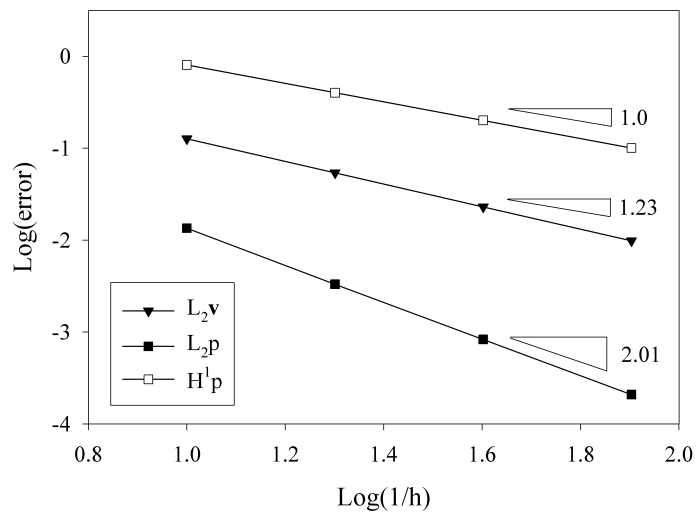
ERROR ESTIMATES

If the scalars have local degree ℓ and the vectors have local degree k then we have

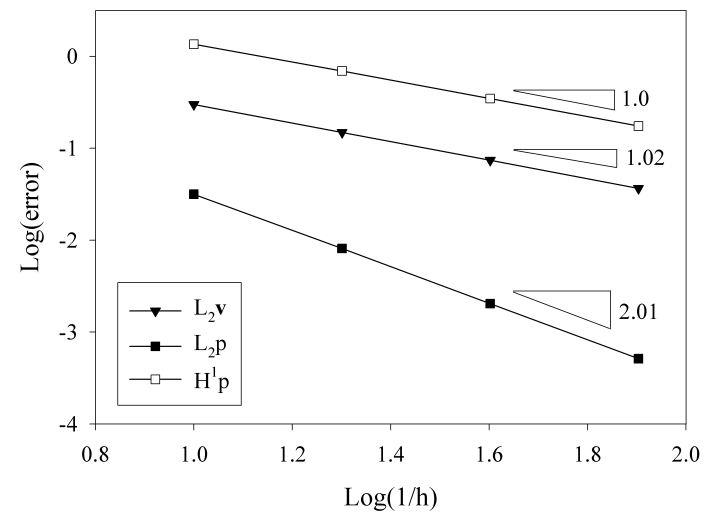
$$|||u - u_h||| \leq C h^s |u|_{s+1, \Omega} \quad \text{with } s := \min\{k + 1, \ell\}$$

$$\|\sigma_h - \sigma\|_{0, \Omega} \leq C |||u_h - u||| \leq C h^s |u|_{s+1, \Omega} \quad \text{with } s := \min\{k + 1, \ell\}$$

Some numerical results (2D)

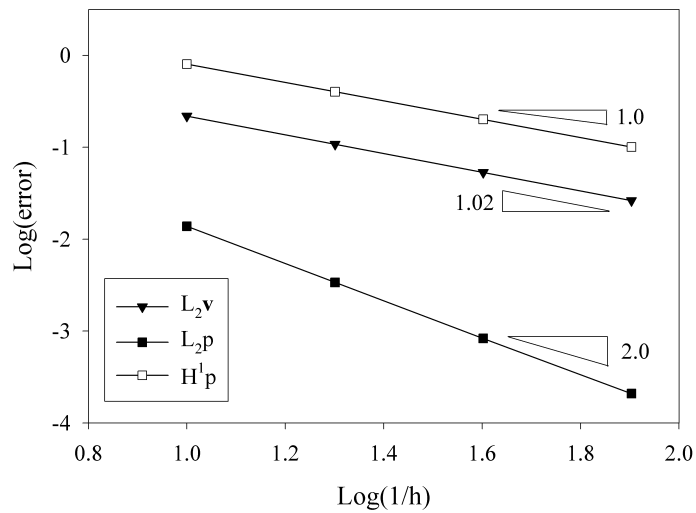


bilinear quads

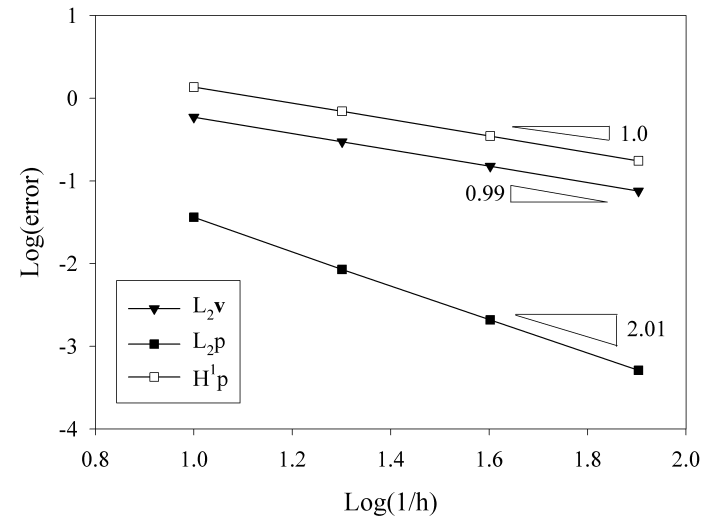


linear triangles

Some numerical results (2D)

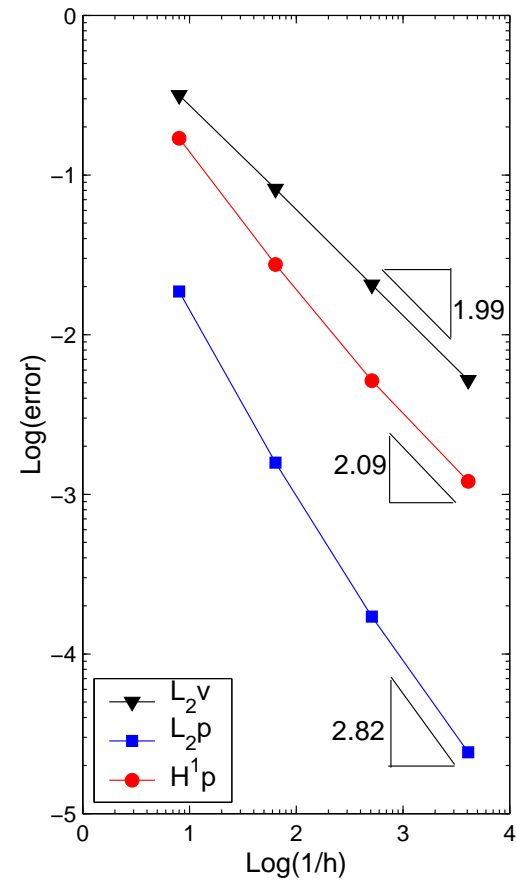
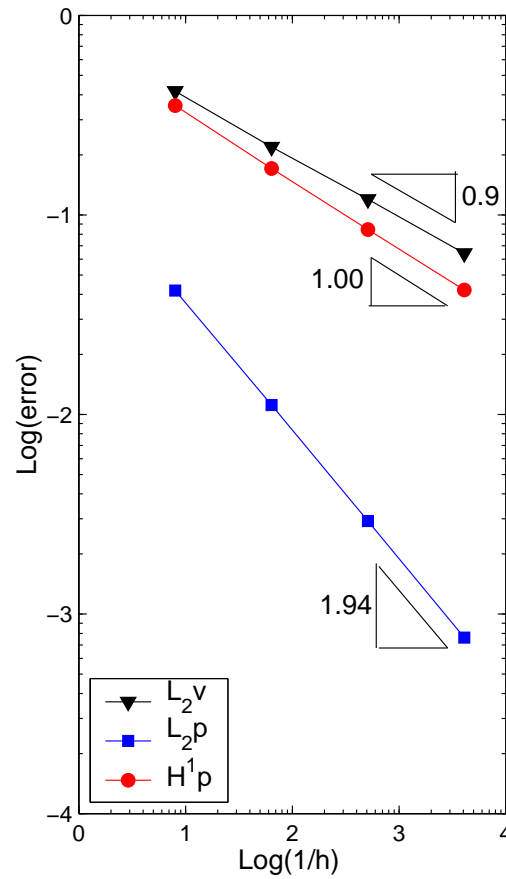


biquadratic σ -bilinear u
quads



quadratic σ -linear u
triangles

Some numerical results (3D) P.Antonietti-L.Heltai

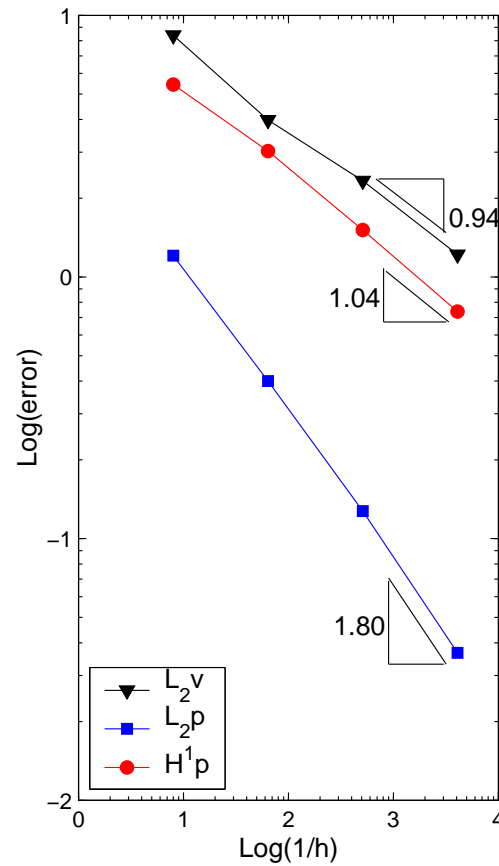


bilinear σ and u

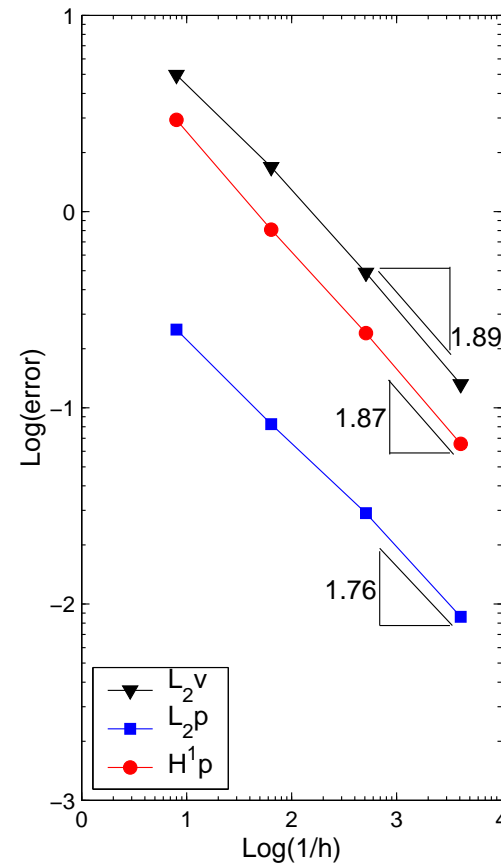
bilinear σ -biquadratic u

Symmetric scheme (Bassi-Rebay + stab with $\theta = 0.5$)

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bilinear σ and u



bilinear σ -biquadratic u

Non symmetric scheme (Bauman-Oden + stab with $\theta = -1$)