Discontinuous Galerkin Methods as Weighted Residuals

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From joint works with

Arnold, Cockburn, Hughes, Marini, Masud, Süli,....

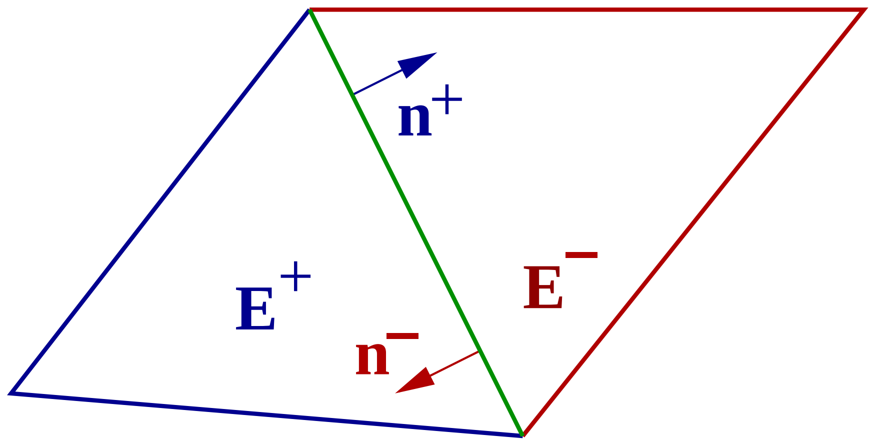
For Michel Crouzeix – Guidel, 2-3 Juin, 2006

PLAN:

- Original Derivation of DG Methods
- The Weighted Residuals Formulation
- Problems in mixed form
- Some numerical Results

AVERAGES AND JUMPS

$\mathscr{T}_h$: decomposition of Ω in elements K; $\mathcal{E}_h$=edges of $\mathscr{T}_h$.



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{ internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{ internal edges}$$

On the boundary edges: $[\![v]\!] = v\mathbf{n}$; $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

A MAGIC FORMULA

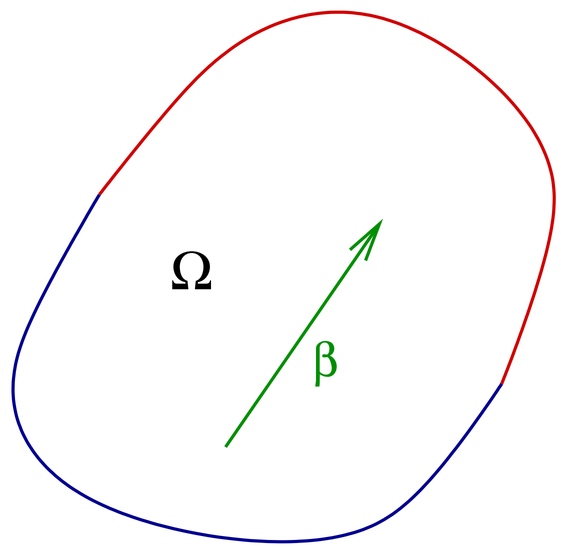
Assume that q is an edge-wise smooth scalar, and $\boldsymbol{\tau}$ an edge-wise smooth vector. We obviously accept that they have different values on the two sides of the same edge. Then, using the above definitions of jump $[\![\cdot]\!]$ and average $\{\cdot\}$, we have

$$\sum_K \int_{\partial K} q\,\boldsymbol{\tau}\cdot\mathbf{n}\,\mathrm{d}s = \sum_e \int_e [\![q]\!]\cdot\{\boldsymbol{\tau}\}\,\mathrm{d}s + \sum_{e'} \int_{e'} \{q\}[\![\boldsymbol{\tau}]\!]\,\mathrm{d}s$$

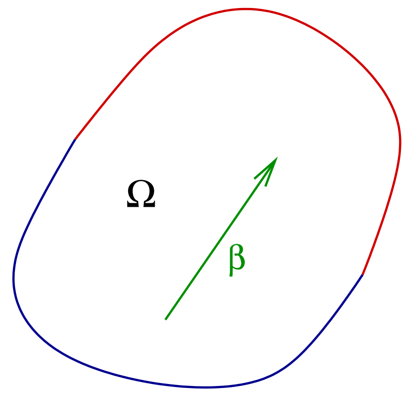
where e ranges over all edges and e' ranges over internal edges.

Clearly, there is **nothing** magic or deep: just *reordering terms in the sum*. But it is surely nice.

A HYPERBOLIC MODEL PROBLEM



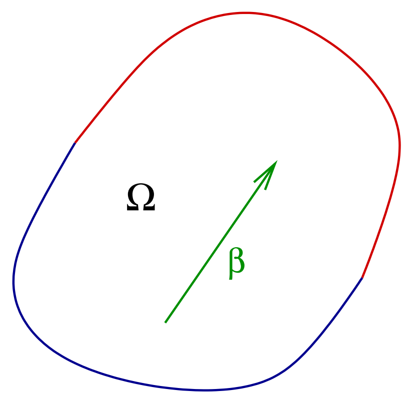
Let Ω be a bounded polygonal domain in $\mathbf{R}^2$, and let the advective velocity field $\boldsymbol{\beta} = (\beta_1,\, \beta_2)^{\mathrm{T}}$ be a vector-valued function defined on $\bar{\Omega}$ with $\beta_i \in C^1(\bar{\Omega}),\; i = 1, 2.$



We define the *inflow* and *outflow* parts of $\Gamma = \partial\Omega$ in the usual fashion:

$$\begin{aligned} \Gamma_- &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) < 0\} = \text{inflow}, \\ \Gamma_+ &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) > 0\} = \text{outflow}, \end{aligned}$$

where $\mathbf{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$.



Let moreover $\gamma \in C(\bar{\Omega})$ be the reactive term, $f \in L^2(\Omega)$ be the external source, and $g \in L^2(\Gamma_-)$ be the Dirichlet datum.

As a model problem we will consider the hyperbolic boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv \operatorname{div}(\boldsymbol{\beta} u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_- \end{aligned} \tag{1}$$

Let us see the DG formulation (Lesaint-Raviart, Reed-Hill).

ORIGINAL DERIVATION OF THE DG METHOD

Multiply the equation by a test function v_h, and integrate over Ω:

$$\int_\Omega (\mathrm{div}(\boldsymbol{\beta} u) + \gamma u) v_h \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x.$$

Then integrate by parts the first term:

$$\sum_{K\in\mathscr{T}_h} \int_K (-u\,(\boldsymbol{\beta}\cdot\nabla v_h) + \gamma u v_h)\mathrm{d}x + \sum_{K\in\mathscr{T}_h} \int_{\partial K} (\boldsymbol{\beta}\cdot\mathbf{n})\,u\,v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Using the magic formula, neglecting $[\![\boldsymbol{\beta} u]\!]$, and using $u = g$ on Γ_-:

$$\sum_{K\in\mathscr{T}_h} \int_K (-u\,(\boldsymbol{\beta}\cdot\nabla v_h) + \gamma u v_h)\mathrm{d}x$$

$$+ \sum_{e\not\subset\Gamma_-} \int_e \{\boldsymbol{\beta} u\}\cdot[\![v_h]\!]\mathrm{d}s + \sum_{e\subset\Gamma_-} \int_e \boldsymbol{\beta}\cdot\mathbf{n}\,g\,v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Then you write the equation putting u_h instead of u

$$\sum_{K\in\mathscr{T}_h}\int_K(-u_h\,(\boldsymbol{\beta}\cdot\nabla v_h)+\gamma u_h v_h)\mathrm{d}x + \sum_{e\not\subset\Gamma_-}\int_e\{\boldsymbol{\beta}u_h\}\cdot[\![v_h]\!]\mathrm{d}s+\sum_{e\subset\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}\,g\,v_h\mathrm{d}s=\int_\Omega f v_h\mathrm{d}x.$$

Finally you substitute the *average* $\{\boldsymbol{\beta}u_h\}$ by the *upwind average* $\{\boldsymbol{\beta}u\}_{up}$, defined, on every edge, as the value of u_h coming from the upwind triangle:

$$\sum_{K\in\mathscr{T}_h}\int_K(-u_h\,(\boldsymbol{\beta}\cdot\nabla v_h)+\gamma u_h v_h)\mathrm{d}x + \sum_{e\not\subset\Gamma_-}\int_e\{\boldsymbol{\beta}u_h\}_{up}\cdot[\![v_h]\!]\mathrm{d}s+\sum_{e\subset\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}\,g\,v_h\mathrm{d}s=\int_\Omega f v_h\mathrm{d}x.$$

AN ELLIPTIC MODEL PROBLEM

Let $\kappa \in \mathcal{L}^\infty(\Omega)$ be the diffusion coefficient, and consider the problem:

$$\begin{cases} A\,u \equiv -\text{div}(\kappa \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

It is often convenient to introduce the flux $\boldsymbol{\sigma} = -\kappa \nabla u$ so that the problem splits in two equations

$$\begin{cases} \kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0 & \text{in } \Omega \\ \text{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Let us see its DG formulation (Arnold, Wheeler, Douglas-Dupont)

You multiply the equation by a test function $v \equiv v_h$ and integrate over Ω. Then integrate by parts.

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v \mathrm{d}x - \sum_K \int_{\partial K} \{\kappa \nabla u\} \cdot \mathbf{n}\, v_h \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

Rearranging terms with the magic formula you have

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} \mathrm{d}s - \sum_{e'} \int_{e'} [\![\kappa \nabla u]\!] \cdot \{v\} \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

Now you say: "Oh, but I know that u is smooth: hence $[\![\kappa \nabla u]\!]$ is zero and I can forget about it!" You then write

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

You had

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa\nabla u\} \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

Then you say: "Since u is smooth, then also $[\![u]\!] = 0$!" And you add a term to restore*symmetry*

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa\nabla u\} \mathrm{d}s - \sum_e \int_e [\![u]\!] \cdot \{\kappa\nabla_h v\} \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

Then you write u_h in place of u

$$\int_\Omega \kappa\nabla_h u_h \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa\nabla_h u_h\} \mathrm{d}s - \sum_e \int_e [\![u_h]\!] \cdot \{\kappa\nabla_h v\} \mathrm{d}s = \int_\Omega f \imath \, \mathrm{d}x.$$

Your discrete problem is now

$$\int_\Omega \kappa \nabla_h u_h \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla_h u_h\} \mathrm{d}s - \sum_e \int_e [\![u_h]\!] \cdot \{\kappa \nabla_h v\} \mathrm{d}s = \int_\Omega f \imath \, \mathrm{d}x.$$

Then you say: "Gosh! My method is unstable! However, since $[\![u]\!] = 0$, I can add a stabilizing term"

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v \mathrm{d}x - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} \mathrm{d}s - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} \mathrm{d}s$$
$$+ \sum_e \frac{\gamma}{|e|} \int_e [\![u_h]\!] \cdot [\![v]\!] \mathrm{d}s = \int_\Omega f v \mathrm{d}x.$$

And you are happy. This is "IP" (Interior Penalty).

DG MIXED FORMULATION

Let us see the DG mixed formulation (Bassi-Rebay, Cockburn-Shu).

You multiply the equations $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ and $\mathrm{div}\boldsymbol{\sigma} = f$ by the test functions $\boldsymbol{\tau}$ and v, respectively, and integrate by parts

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \sum_K \int_K u\,\mathrm{div}\,\boldsymbol{\tau}\mathrm{d}x + \sum_K \int_K \hat{u}\,\boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}s = 0$$

$$-\sum_K \int_K \boldsymbol{\sigma}\cdot\nabla v\mathrm{d}x + \sum_K \int_K v\,\hat{\boldsymbol{\sigma}}\cdot\mathbf{n}\mathrm{d}s = \int_\Omega f v$$

where $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are the numerical fluxes meant to approximate $u_{|\partial E}$ and $\boldsymbol{\sigma}_{|\partial E} \equiv -\kappa\nabla u_{|\partial E}$ (respectively), to be modelled later on.

Then you apply the magic formula assuming that $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are single valued

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega u\,\mathrm{div}_h\boldsymbol{\tau}\,\mathrm{d}x + \sum_{e'}\int_{e'} \hat{u}\,[\![\boldsymbol{\tau}]\!]\mathrm{d}s = 0$$

$$-\int_\Omega \boldsymbol{\sigma}\cdot\nabla_h v\,\mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}}\,[\![v]\!]\mathrm{d}s = \int_\Omega fv$$

Then you put $\boldsymbol{\sigma}_h$ in place of $\boldsymbol{\sigma}$ and u_h in place of u, and you integrate by parts back (!) the first equation

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{\tau} + \int_\Omega \nabla_h u_h\cdot\boldsymbol{\tau}\,\mathrm{d}x + \sum_{e'}\int_{e'}\{\hat{u}-u_h\}[\![\boldsymbol{\tau}]\!]\mathrm{d}s + \sum_e\int_e [\![u_h]\!]\{\boldsymbol{\tau}\}\mathrm{d}s = 0$$

$$-\int_\Omega \boldsymbol{\sigma}\cdot\nabla_h v\,\mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}}\,[\![v]\!]\mathrm{d}s = \int_\Omega fv$$

If your choice of $\hat{u}$ depends only on u_h, and if the gradients of the discretized scalars are contained in the space of discretized vectors, you can use the first equation

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_\Omega \nabla u_h \cdot \boldsymbol{\tau} \mathrm{d}x + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\boldsymbol{\tau}]\!] \mathrm{d}s - \sum_e \int_e [\![u_h]\!]\{\boldsymbol{\tau}\} \mathrm{d}s = 0$$

to express $\boldsymbol{\sigma}_h$ directly as an explicit function of u_h

$$\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h(u_h)$$

and substitute in the second

$$-\int_\Omega \boldsymbol{\sigma}(u_h) \cdot \nabla_h v \mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}}\, [\![v]\!] \mathrm{d}s = \int_\Omega f v.$$

For various choices of $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ you get a whole ZOO of methods (Arnold-B-Cockburn-Marini).

THE NEW APPROACH

(B-Cockburn-Marini-Süli)

Let us see the basic ideas behind it. We consider a rather general system of PDE equations in a domain Ω and on its boundary $\partial\Omega$, that we write as

$$\mathcal{A}(\mathbf{u}) = 0.$$

Here the (nonlinear) operator $\mathcal{A}$ includes the right-hand side as well as the boundary conditions, so that, say, the problem

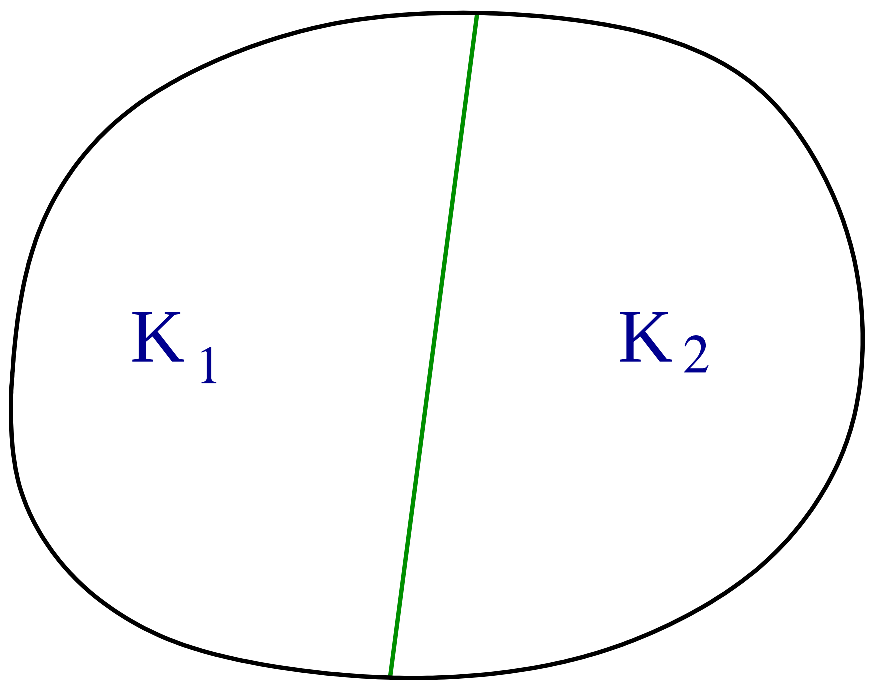
$$-\Delta u = f \text{ in } \Omega \qquad \text{with} \qquad u = g \text{ on } \partial\Omega,$$

would correspond to the operator

$$\mathcal{A}(u) := (\Delta u + f, \gamma_0 u - g)$$

in suitable functional spaces, where γ_0 is the *trace operator* on $\partial\Omega$.

Assume now that you look, *a priori* for a solution $\mathbf{u}$ that is (reasonably) smooth separately in two (or more) subdomains $K_1, ..., K_k$ with $\Omega = \overline{\cup_j \overset{\circ}{K_j}}$.



If we want to write that such a $\mathbf{u}$ is a solution of our original problem $\mathcal{A}(\mathbf{u}) = 0$, we have now to take into account:

i) the residual of the partial differential equation(s) in each element,

ii) the residual in the boundary conditions on $\partial\Omega$, *and*

iii) the jumps of suitable trace operators applied to $\mathbf{u}$ at interfaces (that we consider as residuals of suitable *continuity conditions*).

We indicate all these residuals by $R_1(\mathbf{u}), R_2(\mathbf{u}), ..., R_n(\mathbf{u})$. The original problem $\mathcal{A}(\mathbf{u}) = 0$ is then changed into

$$R_i(\mathbf{u}) = 0 \qquad (i = 1, ..., n)$$

For instance, for Laplace operator in two subdomains we have

$$\Delta u + f = 0 \text{ in } \Omega_i \qquad \gamma_0 u - g = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \qquad (i = 1, 2)$$

$$[\![u]\!] = 0 \qquad [\![\nabla_h u]\!] = 0 \qquad \text{on } \partial\Omega_1 \cap \partial\Omega_2$$

We consider now a decomposition of Ω into polygonal *elements* K_j. In order to proceed toward a discretized problem we would like to impose $R_i(\mathbf{u}) = 0, \quad i = 1, 2, ..., n$ in some weak (variational) sense.

For this we consider a functional space $\mathcal{V}$ with the following properties.

- The functions in $\mathcal{V}$ can be discontinuous from one element K_j to another.
- The residual operators $R_1(\mathbf{u}), R_2(\mathbf{u}), ..., R_n(\mathbf{u})$ map $\mathcal{V}$ either into $L^2(K_j)$ (for some element K_j) or to $L^2(e)$ (for some edge e).
- The exact solution $\mathbf{u}$ belongs to $\mathcal{V}$.

Then we define suitable weight operators $W_1(\mathbf{v}), W_2(\mathbf{v}), ..., W_n(\mathbf{v})$ and we consider the variational problem :

Find $\mathbf{u} \in \mathcal{V}$ *such that*

$$\sum_{i=1}^{n} (R_i(\mathbf{u}), W_i(\mathbf{v}))_i = 0 \qquad \forall\, \mathbf{v} \in \mathcal{V},$$

where the $(.\,,\,.)_i$'s are suitable (L^2-type) inner products (either inside each element, or on the boundaries of the elements). A sufficient condition for having a unique solution is that the $W_i's$ map $\mathcal{V}$ into a dense subset of the corresponding product of L^2 spaces (on elements, and on edges). We shall assume that this is verified.

In particular, you cannot allow any of the W_i to be zero.

We choose now a finite element space V_h of piecewise polynomial functions. If $\mathcal{V}$ is not *totally crazy*, we will have $V_h \subset \mathcal{V}$.

The corresponding discrete problem will than be:

Find $\mathbf{u}_h \in V_h$ *such that*

$$\sum_{i=1}^{n} (R_i(\mathbf{u}_h), W_i(\mathbf{v}_h))_i = 0 \qquad \forall\, \mathbf{v}_h \in V_h.$$

We point out explicitly that the existence and uniqueness that we have for the continuous problem will **not** be inherited by the discrete problem, unless the choices of the weights W_i and of the space V_h are *smart enough*. No *free lunch*. Sorry!.

This approach is quite general and it applies to a number of different problems. It is clear, however, that the stability and the accuracy of the resulting scheme will depend heavily on the choice of the weights $W_1(\mathbf{v}_h), W_2(\mathbf{v}_h), ..., W_n(\mathbf{v}_h)$. Suitable choices of the weights can also help in getting a *nicer* formulation (e.g. symmetric, when the problem is itself symmetric, easier to implement, and so on). However in order to reach a stable method it is, in general, necessary to add a least-square part to them. Roughly speaking, this means choosing *at least one* of the W_i's of the form

$$W_i(\mathbf{v}_h) = N_i(\mathbf{v}_h) + \mu_i R_i(\mathbf{v}_h)$$

where $N_i(\mathbf{v}_h)$ (possibly equal to zero) is the part used to make the method *nicer*, while μ_i is a suitable (in general, mesh dependent) coefficient, and the term $\mu_i R_i(\mathbf{v}_h)$ is there to *stabilize* the method.

WEIGHTED RESIDUALS FORMULATION FOR DARCY

Coming back to the Darcy problem $-\operatorname{div}\kappa\nabla u = f$, with zero Dirichlet boundary conditions, our residual equations are now:

- $-\operatorname{div}\kappa\nabla u - f = 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\boldsymbol{\sigma}]\!] = 0$ on each internal edge

(remember that $\boldsymbol{\sigma} = -\kappa\nabla u$).

A HOMEMADE ESTIMATE

It is not difficult to see that the residual equations above are *equivalent* to the original problem.

For instance if w is a (discontinuous) function, belonging to $H^2(K)$ for each K, then (setting $Aw := -\operatorname{div} \kappa \nabla w$)

$$\|w\|^2_{L^2(\Omega)} \leq C\Big(\sum_{K \in \mathscr{T}_h} \|Aw\|^2_{L^2(K)} + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|[\![w]\!]\|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h^\circ} \frac{1}{|e|} \|[\![\kappa \nabla w]\!]\|^2_{L^2 \ e)} \Big).$$

But, surely, the ”a posteriori guys” know better ;-))

THE STARTING POINT

We choose $\mathcal{V} = H^2(\mathscr{T}_h)$ as the set of functions that are in H^2 of each element, separately. We group the W_i in three operators B_0, B_1, B_2 from $H^2(\mathscr{T}_h)$ to $L^2(\mathscr{T}_h)$, $L^2(\mathcal{E}_h)$ and $L^2(\mathcal{E}_h^\circ)$ respectively.

Then we consider the following *variational* formulation

Find $u \in H^2(\mathscr{T}_h)$ *such that*

$$(A\,u - f,\, B_0\,(v))_h + <[\![u]\!],\, B_1(v)>_h + <[\![\boldsymbol{\sigma}]\!],\, B_2(v)>_h^0 = 0$$

for all $v \in H^2(\mathscr{T}_h)$, where

$$(u,\, v)_h = \sum_{K \in \mathscr{T}_h} \int_K u\, v\, \mathrm{d}x \qquad <u,\, v>_h = \sum_{e \in \mathcal{E}_h} \int_e u\, v\, \mathrm{d}s$$

and $<u,\, v>_h^0$ runs only on internal edges

SUFFICIENT CONDITIONS ON THE OPERATORS B_j

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v \in \mathcal{V}$$

The above equation gives back the original three equations on u (that is: $A\,u = f$, $[\![u]\!] = 0$, and $[\![\boldsymbol{\sigma}]\!] = 0$) **if**

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \boldsymbol{\psi} \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_1 v = \boldsymbol{\psi} \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

CHOICE OF B_0

- Choosing $B_0 v \equiv v$ and using the magic formula, we can write:

$$(Au, B_0 v)_h \equiv \sum_{K \in \mathscr{T}_h} \int_K -\mathrm{div}(\kappa \nabla u)\, v \, \mathrm{d}x$$

$$= \sum_{K \in \mathscr{T}_h} \{ \int_K \kappa \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\partial K} \boldsymbol{\sigma} \cdot \mathbf{n}_K \; v \, \mathrm{d}s \}$$

$$= \sum_{K \in \mathscr{T}_h} \int_K \kappa \nabla u \cdot \nabla v \, \mathrm{d}x + \sum_e \int_e [\![v]\!] \cdot \{ \boldsymbol{\sigma} \} \mathrm{d}s + \sum_{e'} \int_{e'} \{ v \} [\![\boldsymbol{\sigma}]\!] \, \mathrm{d}s$$

$$= (\kappa \nabla u, \; \nabla v)_h + < \{ \boldsymbol{\sigma} \}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{ v \} >_h^0.$$

CHOICE OF B_2

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v \in \mathcal{V}$$

With the previous choice of B_0 we had

$$(Au, B_0 v)_h = (\kappa \nabla u,\, \nabla v)_h + < \{\boldsymbol{\sigma}\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0$$

- Choosing $B_2(v) \equiv -\{v\}$ gives

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0$$
$$\equiv (\kappa \nabla u,\, \nabla\, v)_h - < \{\kappa \nabla u\}, [\![v]\!] >_h - (f, v)_h + < [\![u]\!],\, B_1(v) >_h$$

and we have just to choose B_1

CHOICES FOR B_1

Our equations are

$$(\kappa\nabla u,\, \nabla\, v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h + < [\![u]\!],\, B_1(v) >_h = (f, v)_h$$

You are not allowed to take $B_1 \equiv 0$. But you can take, for instance

- $B_1(v)_{|e} := -\{\kappa\nabla_h\, v\} + \left(\frac{c}{|e|}[\![v]\!]\right)$ (stabilized) IP

or

- $B_1(v)_{|e} := +\{\kappa\nabla_h\, v\} + \left(\frac{c}{|e|}[\![v]\!]\right)$ (stabilized) Baumann-Oden

or

- $B_1(v)_{|e} := \frac{c}{|e|}[\![v]\!]$ Wheeler, Sun

OTHER POSSIBILITIES

On the other hand, the choice of B_2 can also be revisited. For instance taking

$$B_2 v_{|e} = -\{v\} + \gamma|e|[\![\nabla_h v]\!] \quad \text{and} \quad B_1(v)_{|e} := -\{\kappa\nabla_h v\} + \frac{c(\kappa)}{|e|}[\![v]\!]$$

(Douglas-Dupont, Hansbo et als, etc.) your final bilinear form will be

$$(\kappa\nabla u, \nabla v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h - < \{\kappa\nabla_h v\}, [\![u]\!] >_h$$
$$+ \sum_{e\in\mathcal{E}_h} \frac{c(\kappa)}{|e|}\int_e [\![u]\!]\cdot[\![v]\!]\mathrm{d}s + \sum_{e\in\mathcal{E}_h^\circ} \gamma|e|\int_e [\![\kappa\nabla u]\!]\cdot[\![\nabla_h v]\!]\mathrm{d}s$$

ERROR ESTIMATES

$$V_h := \{v \in L^2(\Omega) : \; v_{|K} \in P_r(K) \; \forall K \in \mathscr{T}_h\} \subset H^2(\mathscr{T}_h)$$

The final bilinear form $a(u,v)$ should satisfy:

1) $a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$ 2) $\alpha |||v_h|||^2 \le a(v_h, v_h) \quad \forall v_h \in V_h$

3) $a(w,v) \le C |||w||| \, |||v||| \quad \forall w, v \in H^2(\mathscr{T}_h)$ 4) $|||u - u_I||| \le C\, h^r \, \|u\|_{r+1}$

where

$$|||v|||^2 := \|\nabla_h v\|_0^2 + \sum_e |e|^{-1} \|[\![v]\!]\|_{0,e}^2 + \sum_K h_K^2 |v|_{2,K}^2$$

and u_h (discrete solution), and u_I (interpolant of u) are in V_h. Then:

$$\alpha |||u_h - u_I|||^2 \le a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I)$$

$$\le C \, |||u - u_I||| \, |||u_h - u_I||| \le C\, h^r \, |||u_h - u_I|||.$$

MIXED DG FORMULATIONS FOR DARCY FLOWS

We consider again the mixed formulation for Darcy flow problem, this time with Neumann Boundary conditions

$$\begin{cases} \boldsymbol{\sigma} = -\kappa \nabla u & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = g & \text{on } \Gamma = \partial\Omega. \end{cases}$$

Note that now we have Neumann boundary conditions all over the boundary, instead of the previous Dirichlet ones. Hence we obviously have to require the compatibility condition:

$$\int_\Omega f \,\mathrm{d}x = \int_\Gamma g \,\mathrm{d}s$$

that will give us a unique solution $u \in H^1(\Omega)/\mathbb{R}$, (and a unique $\boldsymbol{\sigma} \in H(\operatorname{div};\Omega)$).

THE DG RESIDUALS FOR MIXED FORMULATIONS

In view of using discontinuous elements, we have to enforce the following equations:

- $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ in each element
- $\operatorname{div} \boldsymbol{\sigma} = f$ in each element
- $[\![u]\!] = 0$ on each internal edge
- $[\![\boldsymbol{\sigma}]\!] = 0$ on each internal edge
- $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on each boundary edge

COUPLING THE RESIDUALS IN TWO EQUATIONS

As we have two unknowns, it will be very convenient to group all the residuals in two equations. This can obviously be done in a number of ways.

For the sake of simplicity, we shall present just **one** way of grouping them, and to make thing further simple, we do it directly at the discrete level.

For this, for every decomposition $\mathscr{T}_h$ of Ω into elements K, and for every couple of nonnegative integers ℓ and k, we introduce the finite element spaces

$$\boldsymbol{\Sigma}_h^k = \{\boldsymbol{\tau} \in [L^2(\Omega)]^2 : \ \boldsymbol{\tau}_{|K} \in [P_k(K)]^2 \quad \forall K \in \mathscr{T}_h\}$$

$$V_h^\ell = \{v \in L^2(\Omega)/\mathbb{R} : \ v_{|K} \in P_\ell(K) \quad \forall K \in \mathscr{T}_h\}.$$

A REASONABLE COUPLING

We first consider the first discretized equation, where we group: $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ in K, $\forall K \in \mathscr{T}_h$ and $[\![u]\!] = 0$ on e, $\forall e \in \mathcal{E}_h^{\circ}$ as follows:

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h))_h + < [\![u_h]\!], B_1(\boldsymbol{\tau}_h) >_h^0 = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k.$$

In the second discretized equation we take $\operatorname{div} \boldsymbol{\sigma} = f$ in K $\forall K \in \mathscr{T}_h$ together with $[\![\boldsymbol{\sigma}]\!] = 0$ on e, $\forall e \in \mathcal{E}_h^{\circ}$, and $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on $\partial\Omega$. We then have:

$$(\operatorname{div}_h \boldsymbol{\sigma}_h - f, B_2(v_h))_h + < [\![\boldsymbol{\sigma}_h]\!], B_3(v_h) >_h^0$$
$$+ < \boldsymbol{\sigma}_h \cdot \mathbf{n} - g, B_3(v_h) >_h^{\partial} = 0 \qquad \forall\ v_h \in V_h^{\ell}$$

A ZOO

Changing the weights, we recover an *incredibly rich* zoo. The analysis, however, follows again *a common pattern*: after eliminating $\boldsymbol{\sigma}_h$ from the first equation one gets

$$B_h(u_h, v_h) = (f, v_h) - < g, v_h > \qquad \forall v_h \in V_h^\ell.$$

Setting $|||v|||^2 := |v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \| [\![v]\!] \|_{0,e}^2$ we check the validity of the classical properties

- Boundedness: very easy in general

  $\exists\ C_b > 0$ such that $B_h(u, v) \leq C_b\, |||u|||\ |||v||| \quad \forall u,\ v \in V(h)$

  where $V(h) := V_h^\ell + H^2(\Omega) \subset H^2(\mathcal{T}_h)$

- Stability: depending on the use of suitable *least square terms*

  $\exists\ C_s > 0$ such that $\quad B_h(v_h, v_h) \geq C_s\, |||v_h|||^2 \qquad \forall v_h \in V_h^\ell$

CONSISTENCY GIVES (MINOR) TROUBLES

Consistency always holds for the original mixed formulations. However, the elimination of $\boldsymbol{\sigma}_h$ from the first equation

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h))_h + < [\![u_h]\!], B_1(\boldsymbol{\tau}_h) >_h^0 = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k,$$

when you plug u in place of u_h, yields $\boldsymbol{\sigma}_h = -\kappa\,\Pi_{\boldsymbol{\Sigma}_h^k}\nabla u$. This, tested against $\nabla_h v_h$, loses the $\Pi_{\boldsymbol{\Sigma}_h^k}$ only if $\nabla_h V_h^\ell \subset \boldsymbol{\Sigma}_h^k$ (that is, with our choices, when $k+1 \geq \ell$).

As a consequence we usually have

$$B_h(u, v_h) = (f, v_h) - \langle g, v_h \rangle$$

only if $\nabla_h V_h^\ell \subset \boldsymbol{\Sigma}_h^k$. Otherwise we have a consistency error which depends on mismatch between the spaces V_h^ℓ and $\boldsymbol{\Sigma}_h^k$. This however can be dealt with rather easily.

BACK TO STABILITY

Stability is not difficult to enforce acting on the jump terms (jump stabilization) *or on the weights inside the elements*, as for instance using a so-called Hughes-Franca stabilization of the type

$$(\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, B_0(\boldsymbol{\tau}_h, v_h))_h := (\kappa^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \boldsymbol{\tau}_h + \theta(\boldsymbol{\tau}_h + \kappa\nabla v_h))_h.$$

It is remarkable that the effects of a Hughes-Franca stabilization are nearly identical to the effects of the jump stabilization. On the other hand, the jump stabilization, in this context, is itself a Hughes - Franca stabilization.

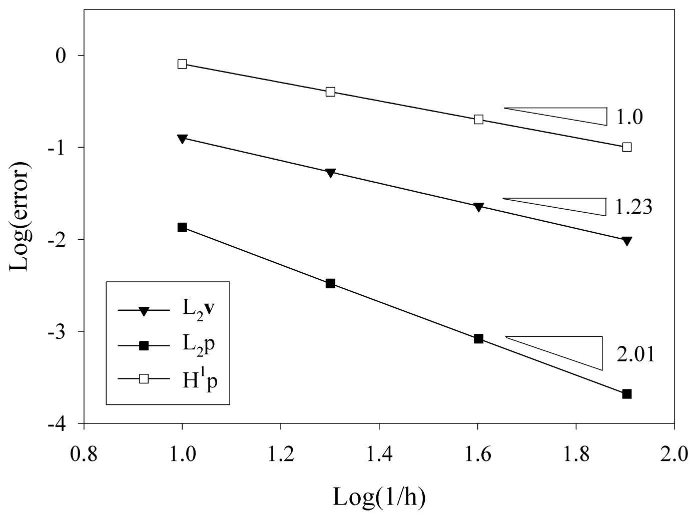
ERROR ESTIMATES

If the scalars have local degree ℓ and the vectors have local degree k then we have

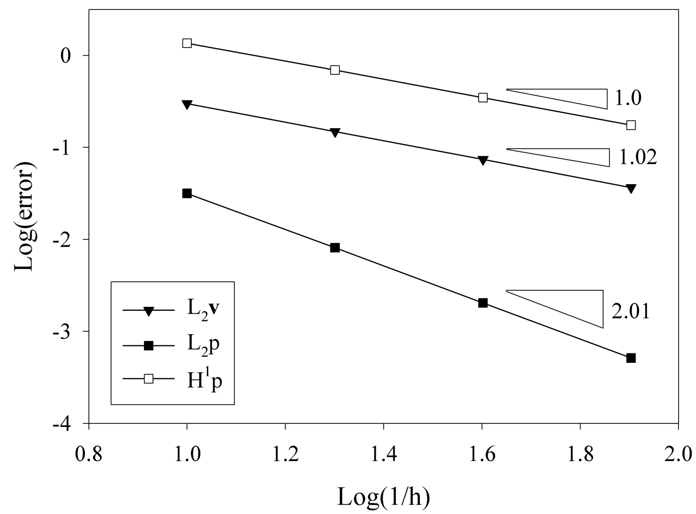
$$|||u - u_h||| \leq C\, h^s |u|_{s+1,\Omega} \quad \text{with } s := \min\{k+1, \ell\}$$

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{0,\Omega} \leq C\, |||u_h - u||| \leq C\, h^s |u|_{s+1,\Omega} \quad \text{with } s := \min\{k+1, \ell\}$$

Some numerical results (2D)

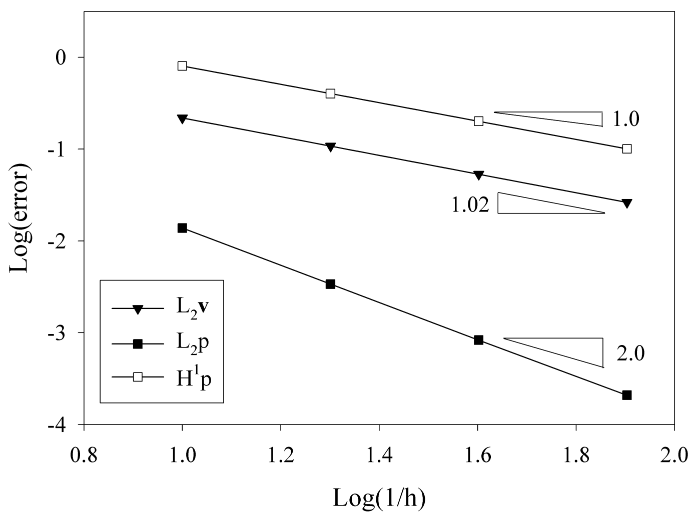


bilinear quads



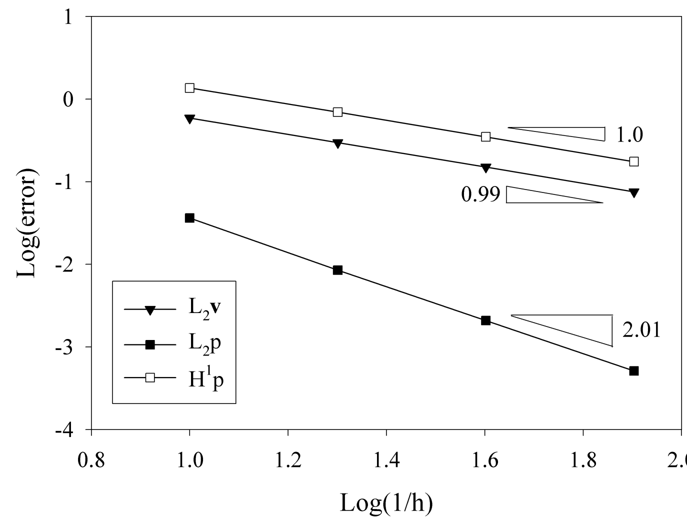
linear triangles

Some numerical results (2D)



biquadratic $\boldsymbol{\sigma}$-bilinear u

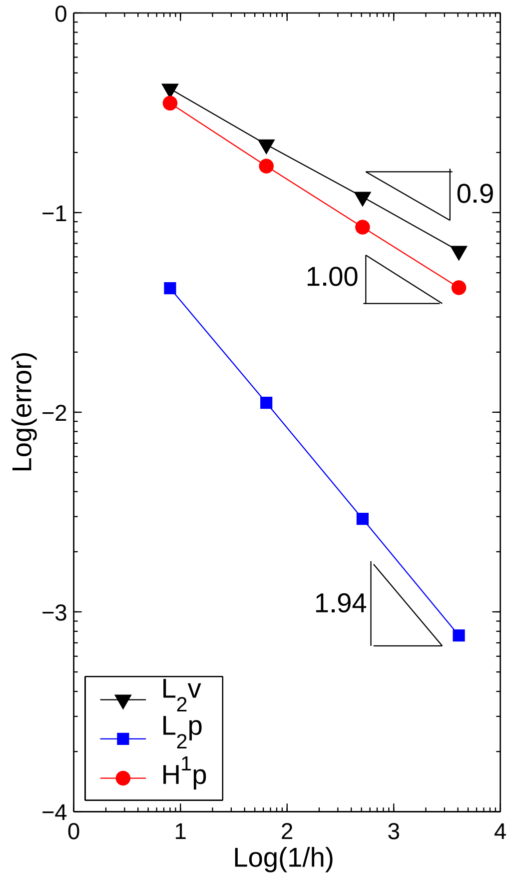
quads

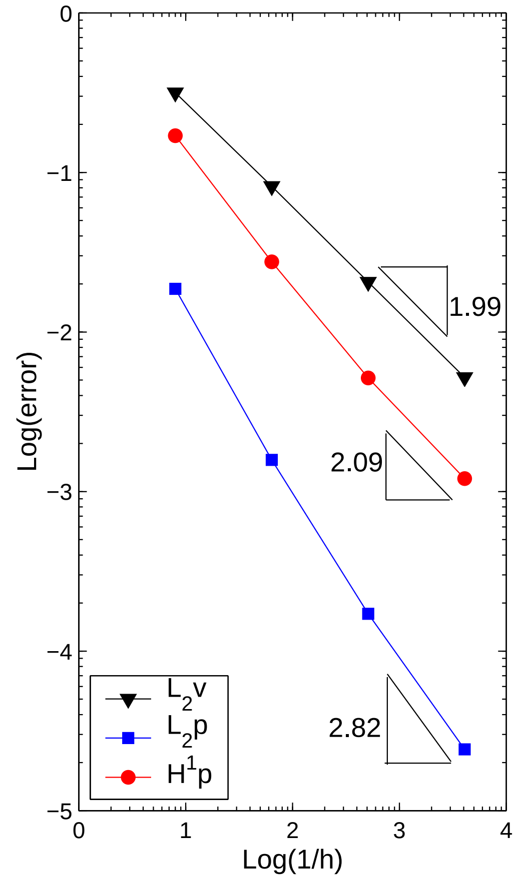


quadratic $\boldsymbol{\sigma}$-linear u

triangles

Some numerical results (3D) P.Antonietti-L.Heltai

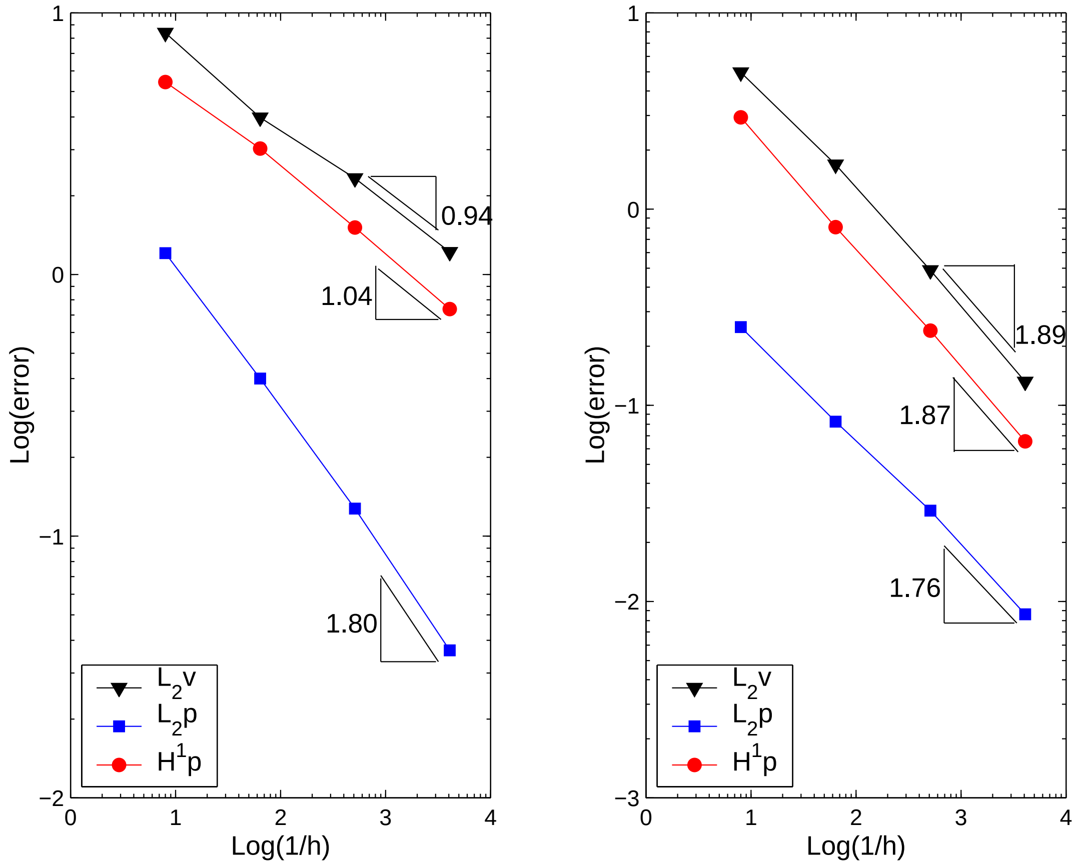




bilinear $\boldsymbol{\sigma}$ and u bilinear $\boldsymbol{\sigma}$-biquadratic u

Symmetric scheme (Bassi-Rebay + stab with $\theta = 0.5$)

Some numerical results (3D) P.Antonietti-L.Heltai



bilinear $\boldsymbol{\sigma}$ and u bilinear $\boldsymbol{\sigma}$-biquadratic u

Non symmetric scheme (Bauman-Oden + stab with $\theta = -1$)